DISSIPATION OF ENERGY DURING MECHANICAL DEFORMATION IN MINERALIZED TISSUE (BONE AS CASE STUDY)

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ABSTRACT

This paper provides a clarified mathematical frame work for studying the energy dissipated during the deformation of Trabecular bone as an example of mineralized tissue. The free energy of the system accounts for both internal energy and deformation. At any point of the deformation, the material is regarded as elastic. The free energy of the system account for both strain and energy dissipated. The exploitation of a dissipation inequality by means provide admissible coupling between stress and energy. Specific admissible constitute equations are derived. The main result of this work is that under pressure but general assumption, the dynamics of dissipation is time regulated by the thermodynamical admissible equation.

Keywords: Dissipation, Energy, Bone, Deformation, Resorption, Thermodynamic, Trabecular.

1.0 INTRODUCTION

The increase in muscle mass due to elevated loads apparently has been known for millennia and the relation between loading and one size has been recognized for centuries (Ascenzi, 1993). Furthermore, more than 100 years ago, Culman and Von Meyer noted a qualitative likeness between the trabecular architecture with femur and principal stress trajectoriesin a similarly shaped crane(Koch,1917). This observation led to Wolff's law or the trajectorial hypothesis for internal bone remodeling (Wolff, 1986).

Remodeling involves changesin material properties. These changes which often are adaptive, may be brought about by alternation in modules, internal structure, strength or density.Using a stress analysis of the femur based on detailed morphological measurement, Koch (1917) confirmed Wolff's law. He also discussed possibility of bones being optimized structures designed for maximum strength with minimum material. Bone and muscle adaptation, however are intimately related since they exert force on each other (Cowin, 1993).

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1.1 STRUCTURE AND MECHANICS OF BONES

Macroscopically, there are two types of bones.

Compact or cortical boneand Cancellous or

trabecular bone. Compact bone is a dense material that makes up primarily the shaft or diaphysis of a long bone. Cancellous bone is porous material that makes up the end or epiphysis of a long bone and is surrounded by thin shell of compact bone. Since the trabecular give Cancellous bone a sponge-like appearance. This type of bone is also called spongy bone.

Microscopically, there are three types of compact bone; Woven, Lamellar, and Harversian (Cowin,1993).Bones contain primarily three types of cells

- 1. Osteoblasts, which create (deposit) bone.
- 2. Osteoclasts, which destroy (resorb) bone.
- 3. Osteocytes, which are cemented osteoblasts that become trapped in the bone matrix and then serve maintenance functions

In this way, osteoclasts first resorb old bone often by tunneling and then osteoblast fill in the tunnels with new bone. The newly deposited bone initially has a relatively low modulus, with mineralization and hardening occurring gradually (David and Andrew, 2012), (Hart and Davy,1989). In this way, deposition occurs without introducing residual stress. Bone resorption and deposition are ongoing process that may be regulated by osteoblasts (Majid, 2010).

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Several mechanisms have been proposed for the transduction of mechanical loads. According to Majid (2010).

- 1. Mechanical fatigue, micro damage
- 2. Alterations in minerals solubility due to stress are parts of mechanical changes in bones. Unlike soft tissues that undergo large deformation, the case of bone is different and therefore the stress, strain and strain-energy criteria seem to have the similar mathematical form. Of these finding non have taken into account the dissipation of energy during mechanical loading. The following significant factors should be considered when developing mathematical model for dissipation of energy in bones or mineralized tissue.
- Different bones have different function and are subjected to different loads, Thus, they are likely to respond differently to the same stress (Khassetarash et al,2015).
- Increase bone mass generally lower stress but increase the weight that must be borne by animal.
- Trabecular bone aligns along the direction of principal stress i.e. Wolff's Law (Hayes and Synder, 1981).
- Trabecular bone density increases in regions of higher shear (Hayes and Synder, 1981) and as the loading rate increases (Khassetarash et al 2015)

2.0 METHODS OF TISSUE DEFORMATION

2.1 BASIC KINEMATICS OF DEFORMED MINERALIZED TISSUE

Initial configuration

Current configuration





We denote the reference configuration by B_r , which is an arbitrary chosen fixed configuration. Then, any particle P of B may be labeled by its position vector X in B_r . Let x be a position vector of P in the current configuration B_t . We say that B occupies the configuration at time t since B_r and B_t are configurations of B, there exist a bijection mapping

$$\chi: B_r \longrightarrow B_t$$
 such that

$$x = \chi_t (X)$$
 for all $X \in B_r$ (1)

 $X = \chi_t^{-1}(x)$ for all $X \in B_t$

The mapping χ is called the deformation of the body from B_r to B_t , since B_t is time dependent, we write

$$\mathbf{x} = \chi_t \left(\mathbf{X} \right) \text{ or } \mathbf{X} = \chi_t^{-1} \left(\mathbf{x} \right) \tag{2}$$

Instead of equation (1), we can have

$$X = \chi (x, t)$$
 for all $X \in B_r$, $t \in I$ (3)

For each particle p (with label x) describes the motion of P with parameter t, and hence the motion of B. it is usual to assume that χ (X, t) is twice continuously differentiable with respect to position and time.

2.2 THE MATERIAL TIME DERIVATIVE

The velocity v of a particle P is defined as

$$v \equiv \dot{x} = \frac{\vartheta}{\vartheta} c(x, t) \tag{4}$$

The acceleration a of P is

$$\mathbf{a} \equiv \dot{v} = \ddot{x} = \frac{\partial^2}{\partial t^2} C(X, t) \tag{5}$$

let ϕ be a scalar field defined on B_t, ie. ϕ , (x, t). since x= $\chi(x, t)$ we may write

$$\phi(\mathbf{x}, t) \ \phi(\chi(\mathbf{X}, t), t) = \phi(\mathbf{X}\mathbf{x} t) \tag{6}$$

Thus any definition on B_r (respectively B_t) can through equation (2) or its inverse, equally be defined on B_t (respectively B_r).

The material derivative of ϕ is the rate of change of ϕ at fixed material point p i.e. at fixed X

Thus, by definition, we have $\dot{\phi} = \frac{\partial}{\partial} f(X, t)$ using chain rule for partial derivatives: we have

$$\frac{\partial}{\partial}f(x,t) = \frac{\partial}{\partial}f(X,t) + \frac{\partial}{\partial}\frac{\partial}{\partial}f(x,t)$$

$$= \frac{\partial}{\partial t} f(x, t) + \frac{\partial}{\partial t} \nabla \phi(x, t).$$

Using equation (4), we have

$$\frac{\partial}{\partial t} f(x,t) \equiv \dot{\phi} = \frac{\partial}{\partial t} f(x,t) \quad (7)$$

Material description

Similarly for vector field,

$$U(x, t) = U(\chi(U, t), t) = U(X, t)$$
 (8)

where in U defined in equation (8), we obtain

spatial description

$$\frac{\partial}{\partial}U(X,t) \equiv \dot{U} = \frac{\partial}{\partial} + (\nabla,\nabla)U$$
 (9)

In particular the acceleration $a = \dot{v}$ is given by

$$a=\dot{v} = \frac{dv}{d} + (v.\nabla)v$$
 (10)

3.0 APPLICATION OF ENERGY BALANCED EQUATIONS TO OBTAIN RESULTS IN DISSIPATION OF ENERGY

3.1 BALANCE EQUATIONS OF MINERALIZED TISSUES (BONES)

In this section we summarize the balance equations, in Lagrangian form that must be satisfied by a dissipation body. Let a material point of the body be labeled by its position vector X in the initial stress – free configuration, denoted B₀. The deformation of the body from B₀ to B_t may then characterized by so-called deformation gradient F, which is defined by $F = \frac{\partial c}{\partial}$ (11)

3.2 ENERGY BALANCE AND CONSTITUTE EQUATIONS

From the principle of conservation of energy,

$$\frac{\partial}{\partial}(K + E) = W + Q \qquad (12)$$

where W is the external force and Q is the mechanical energy that enters the body

In this principle

$$W = \int \rho x_i v_i d + \int T_i v = \int \rho x_i v_i d + \int \delta_i j_i v$$
(13)

and

$$Q = \int_{v} \rho h d - \int_{s} q_{i} n_{i} d \qquad (14)$$

where v is the volume and s, the surface.

Here, we regard W as the mechanical rate of work done on the body of the volume V and the surface S while equation (14) is the total rate of heat input or non-mechanical power, n is the outward unit normal.

Thus according to (Majid, 2010), the principle of conservation of energy (12) reduces to

$$\frac{\partial}{\partial} \int_{V} \rho \left(\frac{1}{2} V_{l} v_{l} + \Sigma \right) d = \int_{V} \rho(x_{l} v_{l} + h) d + \int (\sigma_{l} j_{l} - q_{l}) n$$
(15)

Considering the first integral on the L.H.S. with vowel identity.

$$\frac{\partial}{\partial} \int_{v} \rho \left(\frac{1}{2} v_{i} v_{i} + \Sigma \right) d = \int_{v} \frac{\partial}{\partial} \left[\rho \left(\frac{1}{2} v_{i} v_{i} + \Sigma \right) \right] dv + \left[\rho \left(\frac{1}{2} v_{i} v_{i} + \Sigma \right) v_{j} \right] dv$$

$$\int_{v} \left\{ \frac{u}{D} \left(\frac{1}{2} V_{i} v_{i} + \Sigma \right) \right\} + \left[\rho \left(\frac{1}{2} V_{i} v_{i} + \Sigma \right) \right] V_{j}, j \} d$$
$$= \int_{v} \left\{ \frac{D}{D} + \rho_{j} j \right] \rho \left(\frac{1}{2} V_{i} v_{i} + \Sigma \right) \right\} + \rho (v_{i} d_{i} + \Sigma) \} d$$

= 0 for continuity +
$$\int \rho(v_i a_i + \Sigma) dv =$$

 $\int \rho(v_i a_i + \Sigma) dv$ (16)

The third integral of the equation (15) can be converted to a volume integral by the divergence theorem.

$$\int (\sigma_{t} v_{i} - q_{i}) n_{i} d = \int (\sigma_{t} v_{i}) (17)$$

and hence by substituting of equation (16) and (17) into equation (15) it follows that

$$\int_{v} \{ \rho [v_{l}a_{l} + \Sigma - X_{l}v_{l}v - h] - \sigma_{l-,j}v_{l} - \sigma_{l-,j}v_{l}$$

$$\int_{V} \left\{ V_{l} \left[\rho a_{l} - X_{l} - \sigma_{l,j} \right] + \rho(\Sigma - h) - \sigma_{l} V_{l,j} \right] + q_{l,l} d = 0, \forall n$$
(19)

Since there is no motion, we set

$$\left(V_{i}\sigma_{-i} - X_{i} - \rho_{i,j}\right] = 0 \tag{20}$$

a here
$$\rho \Sigma = \rho h + \sigma_l a_j - q_{l,l}$$
 (21)

Equation (21) is a local form of conservation of energy equation.

For convenience and without loss of generality, we can replace the thermal term in equation (21) by

$$\boldsymbol{\rho} = \boldsymbol{\rho} \boldsymbol{h} - \boldsymbol{q}_{\mathbf{l},j} \tag{22}$$

and
$$a_l = -p\delta_l$$
 (23)

In the local conservation of energy, so that

$$\rho \Sigma = p v_{l,j} + \rho \tag{24}$$

For continuity

$$V_{\mathbf{l},j} = -\frac{1}{\mu} \frac{d}{d} \tag{25}$$

Which reduce equation (20) to

$$\rho \Sigma = \frac{p}{\mu} \frac{d}{d} + \rho \tag{26}$$

Upon setting the specific volume to be $V_s = 1/\rho$, then we obtain the first law of thermodynamics directly from the local conservation of energy equation

Thus,
$$\rho \Sigma = \rho - \frac{p}{v_l} V_s$$
 (27)

That is,

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$$d\Sigma = dq - pd\nu_s \tag{28}$$

 $\boldsymbol{\rho}\mathbf{q}=\mathbf{\rho}\mathbf{h}-\boldsymbol{q}_{\mathbf{I},\mathbf{I}}$

$$p\Sigma = -pv_{\mathbf{i},\mathbf{i}} + \rho$$

$$= -p\boldsymbol{v}_{\mathbf{i},j} + \rho h - \boldsymbol{q}_{\mathbf{i},i} \tag{29}$$

But
$$V_{i,i} = \frac{1}{\rho} \frac{d}{d} + \rho h - q_{l,j}$$
 (30)

$$\therefore \rho.\Sigma = +\frac{p}{\mu}\frac{d}{d} + \rho h - q_{j,l}$$
(31)

Defining $q_I = -k \theta_i$ (32)

$$\rho\Sigma = \frac{p}{\mu}\frac{d}{d} + \rho h + \left(k\frac{d}{d}\theta_{\theta,t}\right), t$$
(33)

$$\frac{d}{d}\left[k+E\right] = \frac{d}{d} + \frac{d}{d} = W + Q \qquad (34)$$

For quasi-static approximation of irreversible processes, and of the thermostatic changes in the continuous system is time independent then,

$$\dot{K} = 0$$

 $\dot{E} dt = dE$ (35)
 $Wdt = \bar{d}w$
 $Qdt = \bar{d} Q$

Where \bar{d} denote an inexact differential, following standard argument above, we have

$$d\mathbf{E} = \bar{\boldsymbol{d}}\mathbf{w} + \bar{\boldsymbol{d}}\mathbf{Q} \tag{36}$$

To convert inexact equation (36) into exact, we use integrating factor 1/T and write the total differential as

$$ds = \frac{\bar{a}Q}{T}$$
(37)

When s is the entropy

Using clausius-duhem inequality

$$S_{B^-} S_A \ge \int_A^M \frac{d}{\tau} \tag{38}$$

This is an equation of irreversible process from an equilibrium state A to and ending in an equilibrium state B. Here we look at some basic assumption such as;

- a) At any point of deformation, the material remains elastic throughout the range of the deformation.
- b) The material is assumed to be homogenous.
- c) The bone structure is considered as a continuous body subjected as a portion of its boundary ∂B ⊂ B to the chemical activity generated by osteoclasts, the biological generated activity leads to an overall change of mass of the solid, presented here by only mineral fraction given by (Emilio et al ,2010)

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(46)

$$\frac{d}{d}\int_{B}\rho = \int_{\sigma}\rho d \qquad (39)$$

Were ρ is the apparent density of bone mineral, which is assumed to remain constant during the resorption process in the remaining ultrastructure B; V= v.n. with convention, v.n=0 correspond to dissipation process provided that the only change in mass of the system result from the molar flux of the dissolved bone matter.

Having establish above for a resorption process we prove the Clausius Duhem in equality, which state that the externally supplied work is dissipated dW_{ext} , when not stored as free dW in the system dissipated into heat form

$$\frac{d}{d} = \frac{dW_e}{d} - \frac{d}{d} \ge 0 \tag{40}$$

Proof

We begin with Clausius – Duhem inequality in integral form as follows

$$\frac{d}{d} \left(\int_{\Omega} \rho \right) \geq \int_{\sigma \Omega} \rho \left(U_n - V . n \right) d - \int_{\Omega} \frac{\rho}{l} d$$
(41)

Can be written in differential form as

$$\rho \geq -\nabla\left(\frac{q}{\iota}\right) + \left(\frac{\rho}{\iota}\right) \tag{42}$$

When η =entropy, ρ = density, q= heat flux, s= energy source, T= temperature

Assuming that Ω is an arbitrary fixed control volume, then $U_n = 0$ and the derivative can be taken inside the integral to give

$$\int \frac{d}{d} (\rho r_i) d \geq -\int \rho (v) d - \int \frac{q \cdot n}{i} d + \int_{\Omega} \frac{\rho}{i} d \qquad (43)$$

Using divergence theorem, we get

$$\int_{\Omega} \frac{d}{d} (\rho) d \ge -\int_{\Omega} \nabla \left(\frac{q}{r}\right) d + \int \frac{\rho}{r} d \qquad (44)$$

Since Ω is arbitrary, we must have

$$\frac{\sigma}{\sigma}(\rho_{-}) \ge -\nabla_{-}(\rho_{-}) - \nabla_{-}\left(\frac{q}{l}\right) + \frac{\rho_{-}}{l}$$
(45)
Expanding out

$$\frac{\sigma}{\sigma}r_{i} + \frac{\rho}{\sigma} \geq -\nabla(\rho) \quad v - \rho \quad (\nabla, v) - \nabla \cdot \left(\frac{q}{l}\right) + \frac{\rho}{l}$$

Or

$$\frac{\sigma}{\sigma}\eta + \frac{\rho}{\sigma} \ge -\eta \nabla \rho. \nu - \rho \nabla \eta. \nu - \rho \quad (\nabla. \nu) - \nabla \eta. \nu = \rho \quad (\nabla. \nu) = 0$$

$$\nabla \cdot \left(\frac{q}{r}\right) + \frac{\rho}{r} \quad (47)$$

$$I\left(\frac{\sigma}{\sigma} + \nabla \rho. v + \rho \nabla. v\right) \eta + \rho \left(\frac{\sigma}{\sigma} + \nabla \eta. v\right) - \nabla. \left(\frac{q}{r}\right) + \frac{\rho}{r}$$
(48)

So, the materials time derivative of $\rho a \eta a g b$

$$\dot{\rho} = \frac{\sigma}{\sigma} + \nabla \rho. \, \nu, \dot{\eta} = \frac{\sigma}{\sigma} \eta + \nabla \eta + \nabla \eta. \, v \tag{49}$$

Therefore

$$(\dot{\rho} + \rho \nabla . \nu)\eta + \rho \dot{\eta} \ge -\nabla . \left(\frac{q}{\iota}\right) + \frac{\rho}{\iota}$$
(50)

$$(\dot{\rho} + \rho \nabla, \nu)\eta = 0, e$$
 (50)g
 $\rho \dot{\eta} \ge -\nabla \left(\frac{q}{l}\right) + \frac{\rho}{l}$ (51)

In terms of internal energy equation (51) can be written as

$$(\rho(\dot{e} - T\dot{\eta}) - \sigma: \nabla \nu \le -\frac{q \cdot v}{T}$$
(52)

Where,

$$\dot{e} = \frac{\sigma}{\sigma} + \nabla e. \nu$$

$$e = e(x, t)$$

$$x = re_r - ze_z \implies r = f(R)$$
(53)

$$\nabla e = \left(\frac{\sigma}{\sigma} + \frac{\sigma}{\sigma} + \frac{\sigma}{\sigma}\right) e. \nu \tag{54}$$

Equation (40) can be redefined when supplied source is absent i.e. S(x, t) = 0 given by

$$\frac{d}{d} = \frac{dW_e}{d} - \frac{d}{d}$$

$$\geq 0 which t: the sc a e \qquad (40)$$

4.0 DISCUSSION OF RESULT

We have established the processes involved in bone formation and the thermodynamics involved in bone deformation. We have also established that externally supplied work to the solid bone, when not stored as free energy is dissipated into heat form. Mathematical Analysis for bone deformation including stress and strain associated with Dissipation of energy in mineralized tissue were established.

5.0 CONCLUSION

The paper gives extensive analysis of dissipation of energy in mineralized tissue with trabecular bone as a case study. Mathematical analysis of energy balance and constitute equation for bone deformation as it affect energy dissipation in mineralized tissue were shown.

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