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ON SOME GENERALIZATIONS OF HARDY-TYPE INTEGRAL INEQUALITIES

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Abstract: In this paper, some new generalizations of Hardy-type integral inequalities were established as an improvement of some known integral inequalities.

1 Introduction

Hardy inequality is one of the classical inequalities which turns information about derivatives of functions into information about the size of the function. This inequality together with its generalizations and extensions are greatly applied in finding solutions to various problems on integral inequalities. It has received the attention of many researchers in the field of analysis as a result of its usefulness in Mathematical analysis.

The discrete form of the inequality was formulated by Hardy [4] in an attempt to find a more simplified proof of the famous Hilbert double series theorem. It is stated in the theorem below.

Theorem 1.1 *If* $p>1$ *and* $\{a_k\}$ *is a sequence of positive real numbers, then*

$$\sum_{n=1}^{\infty}\left[\frac{1}{n}\sum_{k=1}^{n}a_k\right]^p \le C_p \sum_{n=1}^{\infty}a_n^p \tag{1.1}$$

while the integral form of the theorem states that:

Theorem 1.2 *Let* f *be a non-negative p integrable function defined on* $(0,\infty)$ *and* p,∞. *If* f *is integrable over the interval* $(0,x)$, *for each positive* x, *the inequality below is valid.*

$$\int_0^{\infty}\left(\frac{1}{x}\left[\int_0^{x}f(t)\,dt\right]^p\right)dx \le C_p \int_0^{\infty}f^p(x)\,dx \tag{1.2}$$

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In both (1.1) and (1.2), the constant $C_p = \left(\frac{p}{p-1}\right)^p$ is the best possible.

After its introduction and prove, earlier researchers in this area of study focused attention on obtaining a simplified proof of inequality (1.2). For example, Hardy himself in [5] proved that for $p > 1, k \neq 1$ and a function F defined on $\mathbb{R}_+ = (0,\infty)$ by

$$F(x) = \begin{cases} \int_0^x f(t)dt & \text{if } k > 1 \\ \int_x^\infty f(t)dt & \text{if } k < 1 \text{ then,} \end{cases}$$

$$\int_0^\infty x^{-k} F^p(x)\, dx \leq \left[\frac{p}{k-1}\right]^p \int_0^\infty x^{p-k} f^p(x)\, dx \tag{1.3}$$

In another development, Levison in [9] proved that inequality (1.2) holds for $0 < a < b < \infty$. It was Godunova [2] who initiated the direct and simple way of obtaining Hardy inequality via the convexity argument before it was rediscovered by Kaijser 'et 'al [8]. Proving that inequality (1.2) is just a special case of Hardy-Knopp-type inequality given in the theorem below:

Theorem 1.3 *Let φ be a positive and convex function on $(0,\infty)$. Then,*

$$\int_0^\infty \varphi\left(\frac{1}{x}\int_0^x h(t)\, dt\right)^p \frac{dx}{x} \leq \int_0^\infty \varphi(h^p(x))\frac{dx}{x} \tag{1.5}$$

The purpose of this work is to give some Hardy-type inequalities which are generalization and extension of some Hardy-type integral inequalities in literature.

2. Preliminary Lemmas

The following lemmas which are useful in the proof of our main result is presented in this section.

Lemma 2.1 (Chebyshev Integral inequality) [10]:
Let f and g be two functions which are integrable and monotone in the same sense on (a,b) and pe p be a positive and integrable function on the same interval. then,

$$\int_a^b p(x)f(x)dx \int_a^b p(x)g(x)\, dx \leq \int_a^b p(x)f(x)g(x)\, dx \int_a^b p(x)\, dx \tag{2.1}$$

equality occurs if and only if one of the functions f or g reduces to a constant and if f and g are monotone in the opposite sense, the inequality sign are reversed.

Definition 2.1 A function λ is called submultiplicative if $\lambda(xy) \leq \lambda(x)\lambda(y)$ for all

$x, y > 0$ and in particular, if $n \geq 1$, then $\lambda(x^n) \leq \lambda^n(x)$, $x > 0$

3. Main Result

The main results are presented in the theorems below:

Theorem 3.1: Let $f \geq 0$, $g > 0$ non-decreasing on $(0,\infty)$ such that $F(x) := \int_0^x f(t)dt$ and $G(x) := \int_0^x g(t)dt$. If $\varphi \geq 0$ is a function non-decreasing such that $0 < a < b < \infty$ and $\varphi[f(x)g(x)]$ is integrable on $[a,b]$. For $n > 0$, then

$$\int_a^b \varphi\left(\frac{F(x)G(x)}{x^n}\right)dx \leq \int_a^b \varphi\left[x^{-(n-2)} f(x)\, g(x)\right]dx$$

Proof:

By the Chebyshev integral inequality and the assumptions made on the functions f, g and φ. Considering the function $p(x) = 1$ for all $x \in [a,b]$ we have

$$\int_a^b \phi\left(\frac{F(x)G(x)}{x^n}\right)dx \leq \int_a^b \phi\left[\frac{1}{x^n}\left(\int_0^x f(t)dt\right)\left(\int_0^x g(t)dt\right)\right]dx$$

$$\leq \int_a^b \phi\left[x^{-n} \int_0^x dt \int_0^x f(t)g(t)dt\right]dx$$

$$\leq \int_a^b \phi\left[x^{-n} \frac{f(x)}{x} \int_0^x g(t)dt\right]dx$$

$$\leq \int_a^b \phi\left[x^{-n+2} f(x).\, g(x)\right]dx$$

$$\leq \int_a^b \phi\left[\frac{1}{x^{(n-2)}} f(x).\, g(x)\right]dx$$

This ends the proof.

Corollary 3.1:

(i) If in particular we put $n = 2$, we obtain

$$\int_a^b \varphi\left(\frac{F(x)G(x)}{x^2}\right)dx \leq \int_a^b \phi\left[f(x)\, g(x)\right]dx$$

(ii) and if we set $n = 2, g(x) = 1$ and

$$\int_a^b \phi\left[\frac{F(x)}{x}\right]dx \leq \int_a^b \phi(f(x)\, dx)$$

(iii) Taking $\phi(x) = x^p$, we have

$$\int_a^b \left[\frac{F(x)}{x}\right]^p dx \leq \int_a^b f^p(x)\, dx \tag{1.2}$$

The next theorem establishes the converse of Inequality (3.1).

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Theorem 3.2: Let $f \geq 0$, $g > 0$ non-decreasing on $(0,\infty)$ such that $F(x) := \int_0 f(t)dt$ and $G(x) := \int_0 g(t)dt$. If $\varphi \geq 0$ is a function non-increasing such that $0 < a < b < \infty$ and $\varphi[f(x)g(x)]$ is integrable on $[a,b]$. For $n > 0$, then

$$\int \varphi\left(\frac{F(x)G(x)}{x}\right)dx \geq \int \varphi\left[x^{-(\ -2)} f(x)\, g(x)\right]dx$$

Proof:

By the Chebyshev integral inequality and the assumptions made on the functions f, g and φ. Considering the function $p(x) = 1$ for all $x \in [a,b]$ we have

$$\int \varphi\left(\frac{F(x)G(x)}{x}\right)dx = \int \varphi\left[\frac{1}{x}\left(\int_0 f(t)dt\right)\left(\int_0 g(t)dt\right)\right]dx$$

$$\geq \int \varphi\left[x^{-}\ \int_0 dt \int_0 f(t)g(t)dt\right]dx$$

$$\geq \int \varphi\left[x^{-}\ \frac{f(x)}{x}\int_0 g(t)dt\right]dx$$

$$\geq \int \varphi\left[x^{-\ +2} f(x)\,.\,g(x)\right]dx$$

$$= \int \varphi\left[\frac{1}{x^{(\ -2)}} f(x)\,.\,g(x)\right]dx$$

This ends the proof.

Corollary 3.2

(i) If in particular we put $n = 2$, we obtain

$$\int \phi\left(\frac{F(x)G(x)}{x^2}\right)dx \geq \int \phi[f(x)\, g(x)]dx$$

(ii) and if we set $n = 2, \varphi(x) = x^p$, and $g(x) = 1$,, we obtain

$$\int_a^b \left[\frac{F(x)}{x}\right]^p dx \leq \int_a^b f^p(x)\,dx$$

Theorem 3.3: Let $\lambda(x) \geq 0$ be convex, submultiplicative and twice differentiable on $(0,\infty)$. If $\lambda(0) = 0$, $n \in Z$ and $1 < p < \infty$ such that $F(x)$ is defined as $\int_0 f(t)dt$ and $x^{1-}\ \frac{\lambda(xf(x))}{\lambda(x)}$ is integrable while $\frac{\lambda(x)}{x}$ is non-decreasing, then

$$\int_0^\infty \frac{\lambda(x\ F(x))}{\lambda^{\ +2}(x)\, G(x)}\, x^{-\ +1}\, dx \leq \frac{1}{r}\int_0^\infty \lambda(f(t))\, \frac{t^{1-}}{\lambda(t)\, G(t)}\, dt \qquad (3.7)$$

and if in particular $r = p - 1$ then we have

$$\int_0^\infty \frac{\lambda(x^n F(x))}{\lambda^{n+2}(x)\, G(x)} . x^{2-p}\, dx \le \left(\frac{1}{p-1}\right)\int_0^\infty \lambda(f(t))\frac{t^{2-p}}{\lambda(t)\, G(t)}\, dx$$

Proof: The function $\lambda(x)$ is convex, $\frac{\lambda(x)}{x}$ is non-decreasing then by Jensen's inequality, Fubini's theorem and the fact that λ is submultiplicative (in particular $\lambda(x^n) \le \lambda^n(x),\ n \ge 1$), we have

$$\int_0^\infty x^{-\gamma+1} \frac{\lambda(x^n F(x))}{\lambda^{n+2}(x)\, G(x)}\, dx = \int_0^\infty \frac{x^{-\gamma+1}}{\lambda^{n+2}(x)}\, \lambda\left(\frac{x^{n+1} F(x)}{x\, G(x)}\right) dx$$

$$\le \int_0^\infty \frac{x^{-\gamma+1}}{\lambda^{n+2}(x)}\, \lambda(x^{n+1})\lambda\left(\frac{F(x)}{x\, G(x)}\right) dx$$

$$\le \int_0^\infty \frac{x^{-\gamma+1}}{\lambda^{n+2}(x)}\, \lambda^{n+1}(x)\lambda\left(\frac{F(x)}{x\, G(x)}\right) dx$$

$$= \int_0^\infty \frac{x^{-\gamma+1}}{\lambda(x)}\, \lambda\left(\frac{F(x)}{x\, G(x)}\right) dx$$

$$= \int_0^\infty \frac{x^{-\gamma+1}}{\lambda(x)}\, \lambda\left(\frac{1}{x\, G(x)}\int_0^x f(t)dt\right) dx$$

$$= \int_0^\infty \frac{x^{-\gamma+1}}{\lambda(x)}\left(\frac{1}{x\, G(x)}\int_0^x \lambda\, f(t)dt\right) dx$$

$$= \int_0^\infty x^{-\gamma+1}\, \lambda^{-1}(x)\, \frac{1}{x\, G(x)}\left(\int_0^x \lambda\, f(t)dt\right) dx$$

$$= \int_0^\infty \lambda(f(t))\, \frac{t}{\lambda(t)\, G(t)}\left(\int_t^\infty x^{-\gamma-1}\, dx\right) dt$$

$$= \frac{1}{\gamma}\int_0^\infty \lambda(f(t))\, \frac{t}{\lambda(t)\, G(t)}\, t^{-\gamma}\, dt.$$

$$= \frac{1}{\gamma}\int_0^\infty \lambda(f(t))\, \frac{t^{-\gamma+1}}{\lambda(t)\, G(t)}\, dt.$$

which is the required inequality, the proof is complete.
If we put $\gamma = p-1$, we have:

$$\int_0^\infty \frac{\lambda(x^n F(x))}{\lambda^{n+2}(x)\, G(x)} . x^{2-p}\, dx = \frac{1}{p-1}\int_0^\infty \lambda(f(t))\, \frac{t^{2-p}}{\lambda(t)\, G(t)}\, dx$$

Hence inequality (3.7) is a generalization of inequality (5) in [Mehrez 2015] and inequality (2.5) in [Sulaiman 2012].

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