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SOME NEW RESULTS ON HOURGLASS MATRIX

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ABSTRACT
This paper presents some unaddressed issues on establishment of hourglass matrix. An hourglass matrix is a dense square matrix obtained from quadrant interlocking factorization (QIF) of nonsingular matrix. Then, it is shown that every hourglass matrix is invertible and that hourglass matrix (H-matrix) is a subset of Z-matrix.

***Keywords**: Quadrant interlocking factorization, WZ factorization, Z-matrix, Hourglass matrix, Square matrix.*

1.0 INTRODUCTION
Quadrant interlocking factorization (QIF) or WZ factorization was first proposed by Evans and Hatzopoulos (1979) to factorize nonsingular matrix into Z-matrix which was then modified and applied by Evans and Hadjidimos (1980), Evans and Oksa (1997) and Evans (2002). Z-matrix (and also split Z-matrix) is a class of X-matrix (Han & Kye, 2016; Heinig & Rost, 2004). A QIF exists for every nonsingular matrix, often with pivoting - pivoting results in swapping rows or columns in a matrix or by multiplying the matrix with permutation matrices (Babarinsa & Kamarulhaili, 2019; B. Bylina, 2018). The factorization mostly depends on the use of Cramer's rule to know if pivot is required and to check at every stage of the process does not breakdown, see (Babarinsa & Kamarulhaili, 2017). This matrix factorization gives rise to the use of implicit matrix elimination algorithms (that is, Parallel implicit elimination - PIE) for the solution of linear system to simultaneously compute two matrix elements (two columns at a time) for parallel implementation, unlike Gaussian elimination (GE) which computes one column at a time. The factorization seems to be better than the GE and LU factorization irrespective of the number of processors used (D. J. Evans, 1993; D. J. Evans & Abdullah, 1994). To implement these parallel programs on multicore systems with shared-memory, programmers use the OpenMP standard. OpenMP is a standardized set of mechanisms which provides directives and decisions to explicitly define parallel regions (loops) in applications – parallelization (B. Bylina, 2018). In the scientific applications such as mathematical computation, loops are important source of parallelism (J. Bylina & Bylina, 2016).

WZ factorization is known for the adaptability of its direct method to solve system of linear equations. The factorization is the background for the split Levinson-type and the split Schur-type algorithms (Heinig & Rost, 2011). Thus, for the factorization using split Levinson algorithm, split Schur algorithm, ABS and BSP to solve system of linear equations of the form

$$Ax = c, \quad (1)$$

for

$|A| \neq 0,\ A = \{a_{ij}\}\ 1 \leq i,j \leq n,\ \ x = [x_1, x_2, \dots, x_n]^T,\ c = [c_1, c_2, \dots, c_n]^T;\ \ A \in \mathbb{R}^{n \times n},\ c \in \mathbb{R}$

on SIMD or MIMD shared memory parallel computers with multicores such as Intel Xeon

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Phi or IBM's Blue Gene with Many Integrated Core (MIC), see (J. Bylina & Bylina, 2016; D. J. Evans & Barulli, 1998; Golpar-Raboky, 2014; Golpar-Raboky & Mahdavi-Amiri, 2014; Heinig & Rost, 2011). For factorization theorem, Rao (1997) stated that if $A \in \mathbf{R}^{n \times n}$ is a nonsingular matrix that has a unique QIF factorization, then $A = WZ$ if and only if the submatrices of A are invertible. Z-matrix exists together with W-matrix (or a bow-tie matrix) during WZ factorization of nonsingular matrix A (Rhofi & Ameur, 2016), such that

$$A = WZ \tag{2}$$

That is,

$$\overbrace{\begin{bmatrix} a_{1,1} & a_{1,j} & a_{1,n} \\ a_{i,1} & A_{i,j} & a_{i,n} \\ a_{n,1} & a_{n,j} & a_{n,n} \end{bmatrix}}^{A} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ w_{i,1} & W_{n-2} & w_{i,n} \\ 0 & 0 & 1 \end{bmatrix}}^{W} \times \overbrace{\begin{bmatrix} z_{1,1} & z_{1,j} & z_{1,n} \\ 0 & Z_{n-2} & 0 \\ z_{n,1} & z_{n,j} & z_{n,n} \end{bmatrix}}^{Z}.$$

Due to the structure of Z-matrix, many authors had wrongly classified all classes of Z-matrix as hourglass matrix without considering the components of their entries. Unfortunately, there are changes in structure of Z-matrix from QIF which depend on the type of matrix (Toeplitz, Hankel, centrosymmetric, diagonally dominant or tridiagonal matrix) being factorized. Although the term hourglass matrix was first coined by Demeure (1989) in describing the way of factorizing square matrices, especially from real symmetric Toeplitz matrix $\left(T_n = \left[a_{i-j}\right]^{n}_{i,j=1}\right)$ or Hankel matrix $\left(H_n = \left[h_{i+j-1}\right]^{n}_{i,j=1}\right)$, which he later referred that hourglass matrix is synonymous to Z-matrix. The notion or idea of sameness between hourglass matrix and Z-matrix was dropped over time without a cogent reason. In Section (2), we give a brief note on hourglass matrix. While in Section (3), we deduce that every matrix that assume hourglass factorization also assumes WZ factorization, but the converse is false, and that hourglass matrix is invertible. We also give the MATLAB code to generate the random hourglass matrix. Lastly, we give the distinctions between hourglass matrix and Z-matrix to conclude that hourglass matrix is a subset of Z-matrix.

2.0 SHORT NOTES ON HOURGLASS MATRIX

The comprehensive steps to obtain hourglass matrix from QIF known as WH factorization including its MATLAB code, and some deductions about the matrix are detailed in (Babarinsa & Kamarulhaili, 2018). However, the review on hourglass matrix are given in this section to better understanding of the matrix. First, we denote hourglass matrix of order n as H^{*}_{g} (or H-matrix) and its nonzero entries as $h^{*}_{i,j}$. We refer the factorization of hourglass matrix from nonsingular matrix as WH factorization.

Definition 2.1. (Babarinsa & Kamarulhaili, 2018) Let H^{*}_{g} be an hourglass matrix of order $n(n \geq 3)$ with strictly nonzero elements $h^{*}_{i,j} \in \mathbf{R}$, defined as

$$H^{*}_{g} = \begin{cases} h^{*}_{i,j} & 1 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor \quad i \leq j \leq n+1-i; \\ h^{*}_{i,j} & \left\lceil \frac{n+2}{2} \right\rceil \leq i \leq n \quad n+1-i \leq j \leq i; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

The QIF algorithm of Z-matrix is like hourglass matrix except the restriction to produce nonzero elements during the factorization. The QIF of hourglass matrix (WH factorization) from nonsingular matrix computes $w^{*(k)}_{i,k}$ and $w^{*(k)}_{i,n-k+1}$ to generalize for every update of $H^{*(k)}_{g}$ and proceed similarly for the inner square matrices of size $(n-2k)$ and so on by solving 2×2 linear systems

$$\begin{cases} h^{*(k-1)}_{k,k} w^{*(k)}_{i,k} + h^{*(k-1)}_{n-k+1,k} w^{*(k)}_{i,n-k+1} = -h^{*(k-1)}_{i,k} \\ h^{*(k-1)}_{k,n-k+1} w^{*(k)}_{i,k} + h^{*(k-1)}_{n-k+1,n-k+1} w^{*(k)}_{i,n-k+1} = -h^{*(k-1)}_{i,n-k+1} \end{cases} \tag{4}$$

Then, we finally compute for kth steps of $h^{*(k)}_{i,j}$ from Equation (4) as

$$h_{i,j}^{\bullet(k)} = h_{i,j}^{\bullet(k-1)} + w_{i,k}^{\bullet(k)} h_{k,j}^{\bullet(k-1)} + w_{i,n-k+1}^{\bullet(k)} h_{n-k+1,j}^{\bullet(k-1)} \tag{5}$$

For $k = 1,2,\ldots,\frac{n}{2}$; $i,j = k+1,\ldots,n-k$.

The computation of Equation (5) yields hourglass matrix ($H_g^{\bullet(k)} = W^{\bullet(k)} H_g^{\bullet(k-1)} = H_g^{\bullet}$) as

$$H = \begin{bmatrix} h' & h' & h' & \cdots & \cdots & \cdots & \cdots & \cdots & h' & h' & h' \\ 0 & h' & h' & \cdots & \cdots & \cdots & \cdots & \cdots & h' & h' & 0 \\ 0 & 0 & h' & \cdots & \cdots & \cdots & \cdots & \cdots & h' & 0 & 0 \\ \vdots & 0 & 0 & \ddots & \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} & 0 & 0 & \vdots \\ \vdots & \vdots & \vdots & & h' & \cdots & h' & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & & \vdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & h' & \cdots & h' & & \vdots & \vdots & \vdots \\ \vdots & 0 & 0 & \cdot^{\cdot^{\cdot}} & \vdots & \vdots & \vdots & \ddots & 0 & 0 & \vdots \\ 0 & 0 & h' & \cdots & \cdots & \cdots & \cdots & \cdots & h' & 0 & 0 \\ 0 & h' & h' & \cdots & \cdots & \cdots & \cdots & \cdots & h' & h' & 0 \\ h' & h' & h' & \cdots & \cdots & \cdots & \cdots & \cdots & h' & h' & h' \end{bmatrix} \tag{6}$$

If one of the computed entries from Equation (5) is zero, then apply possible row-interchange in no more than $(n-2k)$ times in $H_g^{\bullet(k-1)}$ else the factorization breakdown to produce $H_g^{\bullet}$ (H-matrix). The time complexity of WH factorization is $O(n^4)$ because swapping or sorting rows or columns at every stage in WH factorization increases the computational cost of an algorithm. The rows or columns interchange is necessary for the factorization to work thereby making it numerical stable.

Proposition 2.1. (Babarinsa & Kamarulhaili, 2018) Let $H_g^{\bullet}$, $T_{(nonzero)}$ and $T_{(zero)}$ be an hourglass matrix, the total number of nonzero entries and the total number of zero entries in hourglass matrix respectively. Then

$$T_{(nonzero)} = \frac{1}{2}\left(n^2 + 2n - |(n+1) mod\ 2 - 1|\right)$$

and

$$T_{(zero)} = \frac{1}{2}\left(n^2 - 2n + |(n+1) mod\ 2 - 1|\right).$$

Definition 2.2. (Babarinsa & Kamarulhaili, 2018) Epicenter element, denoted as $h^{\bullet}_{\frac{n+1}{2},\frac{n+1}{2}}$, is the nonzero element located at $\frac{n+1}{2}$ row and $\frac{n+1}{2}$ column of $H_g^{\bullet}$ of odd order.

Definition 2.3. (Babarinsa & Kamarulhaili, 2018) Filanz submatrix, an acronym for first-last nonzero elements of rows in $H_g^{\bullet}$ denoted as $f_m^{1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil}$, is a nonsingular 2×2 submatrix obtained by taking the first and the last nonzero elements of the ith and $(n+1-i)$th row of $H_g^{\bullet}$ as

$$f_m^{1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil} = \begin{bmatrix} h^{\bullet}_{i,i} & h^{\bullet}_{i,n+i-1} \\ h^{\bullet}_{n+i-1,i} & h^{\bullet}_{n+i-1,n+i-1} \end{bmatrix}_{1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil} \tag{7}$$

Proposition 2.2. (Babarinsa & Kamarulhaili, 2018) Given filanz minor $f_m^{1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil}$ of an hourglass matrix $H_g^{\bullet}$ of order $n (n \ge 3)$ \$. Then

$$Det(H_g^{\bullet}) = \begin{cases} \prod_{i=1}^{\left\lceil \frac{n-1}{2} \right\rceil} \left| f_m^{1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil} \right| & if\ n\ is\ e \\ h^{\bullet}_{\frac{n+1}{2},\frac{n+1}{2}} \prod_{i=1}^{\left\lceil \frac{n-1}{2} \right\rceil} \left| f_m^{1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil} \right| & if\ n\ is\ o \end{cases} \tag{8}$$

Proposition 2.3. (Babarinsa & Kamarulhaili, 2018) If n is odd and $|H_g^{\bullet}| \neq 0$, then there exists at least an eigenvalue, $\lambda \in \mathbb{R}$, such that $\lambda = h^{\bullet}_{\frac{n+1}{2},\frac{n+1}{2}}$.

2.1 Numerical Example of WH Factorization

Let us factorize the following 6×6 dense nonsingular square matrix A into hourglass matrix

$$A = \begin{bmatrix} 2 & 0 & 2 & 4 & 3 & -1 \\ 5 & 10 & -7 & 8 & 11 & 4 \\ 0 & -12 & 9 & 6 & 18 & 1 \\ -13 & 12 & 8 & -20 & 14 & 17 \\ 3 & 1 & 1 & -1 & 1 & 4 \\ 10 & 6 & 9 & -13 & 10 & 14 \end{bmatrix}$$

Step 1: We check the first and last row of matrix A before the initial update

$a^{*(0)}_{1.1} = h^{*(0)}_{1.1} = 2, a^{*(0)}_{1.2} = h^{*(0)}_{1.2} = 0,$

$a^{*(0)}_{1.3} = h^{*(0)}_{1.2} = 2, a^{*(0)}_{1.4} = h^{*(0)}_{1.4} = 4,$

$a^{*(0)}_{1.5} = h^{*(0)}_{1.5} = 3, a^{*(0)}_{1.6} = h^{*(0)}_{1.6} = -1$

$a^{*(0)}_{6.1} = h^{*(0)}_{6.1} = 10, a^{*(0)}_{6.2} = h^{*(0)}_{6.2} = 6,$

$a^{*(0)}_{6.3} = h^{*(0)}_{6.2} = 9, a^{*(0)}_{6.4} = h^{*(0)}_{6.4} = -13,$

$a^{*(0)}_{6.5} = h^{*(0)}_{6.5} = 10, a^{*(0)}_{6.6} = h^{*(0)}_{6.6} = 14$

Since $h^{*(0)}_{1.2} = 0$, then we interchange the first row with any other row except the last row. In this case we interchange first row with the fifth row such that the first and last row of the matrix has no zero entry as

$$H^{*(0)} = \begin{bmatrix} 3 & 1 & 1 & -1 & 1 & 4 \\ 5 & 10 & -7 & 8 & 11 & 4 \\ 0 & -12 & 9 & 6 & 18 & 1 \\ -13 & 12 & 8 & -20 & 14 & 17 \\ 2 & 0 & 2 & 4 & 3 & -1 \\ 10 & 6 & 9 & -13 & 10 & 14 \end{bmatrix}$$

With $H^{*(0)}$ having permutation matrix

$$P^{*(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We begin to compute the set of 2×2 system of linear equations from

$$\begin{cases} h^{*(k-1)}_{k,k} w^{*(k)}_{i,k} + h^{*(k-1)}_{n-k+1,k} w^{*(k)}_{i,n-k+1} = -h^{*(k-1)}_{i,k} \\ h^{*(k-1)}_{k,n-k+1} w^{*(k)}_{i,k} + h^{*(k-1)}_{n-k+1,n-k+1} w^{*(k)}_{i,n-k+1} = -h^{*(k-1)}_{i,n-k+1} \end{cases}.$$

For $k = 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor = 2.$

Now, let $k = 1$ then we have

$$\begin{cases} h^{*(0)}_{1.1} w^{*(1)}_{i.1} + h^{*(0)}_{n.1} w^{*(1)}_{i.n} = -h^{*(0)}_{i.1} \\ h^{*(0)}_{1.n} w^{*(1)}_{i.1} + h^{*(0)}_{n.n} w^{*(1)}_{i.n} = -h^{*(0)}_{i.n} \end{cases}$$

Whenever $i = 2$ then

$$\begin{cases} h^{*(0)}_{1.1} w^{*(1)}_{2.1} + h^{*(0)}_{6.1} w^{*(1)}_{2.6} = -h^{*(0)}_{2.1} \\ h^{*(0)}_{1.6} w^{*(1)}_{2.1} + h^{*(0)}_{6.6} w^{*(1)}_{2.6} = -h^{*(0)}_{2.6} \end{cases} \Rightarrow \begin{cases} 3w^{*(1)}_{2.1} + 10w^{*(1)}_{2.6} = -5 \\ 4w^{*(1)}_{2.1} + 14w^{*(1)}_{2.6} = -4 \end{cases} \Rightarrow \begin{cases} w^{*(1)}_{2.1} = -15 \\ w^{*(1)}_{2.6} = 4 \end{cases}$$

Whenever $i = 3$ then

$$\begin{cases} h^{*(0)}_{1.1} w^{*(1)}_{3.1} + h^{*(0)}_{6.1} w^{*(1)}_{3.6} = -h^{*(0)}_{3.1} \\ h^{*(0)}_{1.6} w^{*(1)}_{3.1} + h^{*(0)}_{6.6} w^{*(1)}_{3.6} = -h^{*(0)}_{3.6} \end{cases} \Rightarrow \begin{cases} 3w^{*(1)}_{3.1} + 10w^{*(1)}_{3.6} = 0 \\ 4w^{*(1)}_{3.1} + 14w^{*(1)}_{3.6} = -6 \end{cases} \Rightarrow$$ [illegible]

Whenever $i = 4$ then

$$\begin{cases} h^{*(0)}_{1.1} w^{*(1)}_{4.1} + h^{*(0)}_{6.1} w^{*(1)}_{4.6} = -h^{*(0)}_{4.1} \\ h^{*(0)}_{1.6} w^{*(1)}_{4.1} + h^{*(0)}_{6.6} w^{*(1)}_{4.6} = -h^{*(0)}_{4.6} \end{cases} \Rightarrow \begin{cases} 3w^{*(1)}_{4.1} + 10w^{*(1)}_{4.6} = 13 \\ 4w^{*(1)}_{4.1} + 14w^{*(1)}_{4.6} = -17 \end{cases} \Rightarrow \begin{cases} w^{*(1)}_{4.1} = 176 \\ w^{*(1)}_{4.6} = \frac{-103}{2} \end{cases}$$

Whenever $i = 5$ then

$$\begin{cases} h^{*(0)}_{1.1} w^{*(1)}_{5.1} + h^{*(0)}_{6.1} w^{*(1)}_{5.6} = -h^{*(0)}_{5.1} \\ h^{*(0)}_{1.6} w^{*(1)}_{5.1} + h^{*(0)}_{6.6} w^{*(1)}_{5.6} = -h^{*(0)}_{5.6} \end{cases} \Rightarrow \begin{cases} 3w^{*(1)}_{5.1} + 10w^{*(1)}_{5.6} = -2 \\ 4w^{*(1)}_{5.1} + 14w^{*(1)}_{5.6} = 1 \end{cases} \Rightarrow \begin{cases} w^{*(1)}_{4.1} = -19 \\ w^{*(1)}_{4.6} = \frac{11}{2} \end{cases}$$

Therefore, we write the values of $w^{*(1)}_{i.1}$ and $w^{*(1)}_{i.n}$ in a matrix form as

$$W^{*(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -15 & 1 & 0 & 0 & 0 & 4 \\ 30 & 0 & 1 & 0 & 0 & -9 \\ 176 & 0 & 0 & 1 & 0 & \frac{-103}{2} \\ -19 & 0 & 0 & 0 & 1 & \frac{11}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2: We update $H^{*(0)}$ to $H^{*(1)}$ by computing its entries as

$$h^{*(k)}_{i,j} = h^{*(k-1)}_{i,j} + w^{*(k)}_{i,k} h^{*(k-1)}_{k,j} + w^{*(k)}_{i,n-k+1} h^{*(k-1)}_{n-k+1,j} \Rightarrow h^{*(1)}_{i,j} = h^{*(0)}_{i,j} + w^{*(1)}_{i,1} h^{*(0)}_{1,j} + w^{*(1)}_{i,6} h^{*(0)}_{6,j}$$

When $i = 2$ and $j = 2, 3, 4, 5$ then

$h^{*(1)}_{2.2} = h^{*(0)}_{2.2} + w^{*(1)}_{2.1} h^{*(0)}_{1.2} + w^{*(1)}_{2.6} h^{*(0)}_{6.2} = 10 + (-15)(1) + 4(6) = 19$

$h^{*(1)}_{2.3} = h^{*(0)}_{2.3} + w^{*(1)}_{2.1} h^{*(0)}_{1.3} + w^{*(1)}_{2.6} h^{*(0)}_{6.3} = -7 + (-15)(1) + 4(9) = 14$

$h^{*(1)}_{2.4} = h^{*(0)}_{2.4} + w^{*(1)}_{2.1} h^{*(0)}_{1.4} + w^{*(1)}_{2.6} h^{*(0)}_{6.4} = 8 + (-15)(-1) + 4(-13) = -29$

$h^{*(1)}_{2.5} = h^{*(0)}_{2.5} + w^{*(1)}_{2.1} h^{*(0)}_{1.5} + w^{*(1)}_{2.6} h^{*(0)}_{6.5} = 11 + (-15)(1) + 4(10) = 36$

When $i = 3$ and $j = 2, 3, 4, 5$ then

$h^{*(1)}_{3.2} = h^{*(0)}_{3.2} + w^{*(1)}_{3.1} h^{*(0)}_{1.2} + w^{*(1)}_{3.6} h^{*(0)}_{6.2} = -12 + 30(1) + (-9)(6) = -36$

$h^{*(1)}_{3.3} = h^{*(0)}_{3.3} + w^{*(1)}_{3.1} h^{*(0)}_{1.3} + w^{*(1)}_{3.6} h^{*(0)}_{6.3} = 9 + 30(1) + (-9)(9) = -42$

$h^{*(1)}_{3.4} = h^{*(0)}_{3.4} + w^{*(1)}_{3.1} h^{*(0)}_{1.4} + w^{*(1)}_{3.6} h^{*(0)}_{6.4} 6 + 30(-1) + (-9)(-13) = 93$

$h^{*(1)}_{3.5} = h^{*(0)}_{3.5} + w^{*(1)}_{3.1} h^{*(0)}_{1.5} + w^{*(1)}_{3.6} h^{*(0)}_{6.5} = 18 + 30(1) + (-9)(10) = -42$

When $i = 4$ and $j = 2, 3, 4, 5$ then

$h^{*(1)}_{4.2} = h^{*(0)}_{4.2} + w^{*(1)}_{4.1} h^{*(0)}_{1.2} + w^{*(1)}_{4.6} h^{*(0)}_{6.2} = 12 + 176(1) + \left(\frac{-103}{2}\right)(6) = -121$

$h^{*(1)}_{4.3} = h^{*(0)}_{4.3} + w^{*(1)}_{4.1} h^{*(0)}_{1.3} + w^{*(1)}_{4.6} h^{*(0)}_{6.3} = 8 + 176(1) + \left(\frac{-103}{2}\right)(9) = \frac{-559}{2}$

$h^{*(1)}_{4.4} = h^{*(0)}_{4.4} + w^{*(1)}_{4.1} h^{*(0)}_{1.4} + w^{*(1)}_{4.6} h^{*(0)}_{6.4} = -20 + 176(-1) + \left(\frac{-103}{2}\right)(-13) = \frac{947}{2}$

$h^{*(1)}_{4.5} = h^{*(0)}_{4.5} + w^{*(1)}_{4.1} h^{*(0)}_{1.5} + w^{*(1)}_{4.6} h^{*(0)}_{6.5} = 14 + 176(1) + \left(\frac{-103}{2}\right)(10) = -325$

When $i = 5$ and $j = 2, 3, 4, 5$ then

$h^{*(1)}_{5.2} = h^{*(0)}_{5.2} + w^{*(1)}_{5.1} h^{*(0)}_{1.2} + w^{*(1)}_{5.6} h^{*(0)}_{6.2} = 0 + (-19)(1) + \left(\frac{11}{2}\right)(6) = 14$

$h^{*(1)}_{5.3} = h^{*(0)}_{5.3} + w^{*(1)}_{5.1} h^{*(0)}_{1.3} + w^{*(1)}_{5.6} h^{*(0)}_{6.3} = 2 + (-19)(1) + \left(\frac{11}{2}\right)(9) = \frac{65}{2}$

$h^{*(1)}_{5.4} = h^{*(0)}_{5.4} + w^{*(1)}_{5.1} h^{*(0)}_{1.4} + w^{*(1)}_{5.6} h^{*(0)}_{6.4} = 4 + (-19)(1) + \left(\frac{11}{2}\right)(-13) = \frac{-97}{2}$

$$h^{*(1)}_{5,5} = h^{*(0)}_{5,5} + w^{*(1)}_{5,1} h^{*(0)}_{1,5} + w^{*(1)}_{5,6} h^{*(0)}_{6,5} = 3 + (-19)(1) + \left(\frac{11}{2}\right)(10) = 39$$

Thus,

$$H^{*(1)} = \begin{bmatrix} 3 & 1 & 1 & -1 & 1 & 4 \\ 0 & 19 & 14 & -29 & 36 & 0 \\ 0 & -36 & -42 & 93 & -42 & 0 \\ 0 & -121 & \frac{-559}{2} & \frac{947}{2} & -325 & 0 \\ 0 & 14 & \frac{65}{2} & \frac{-97}{2} & 39 & 0 \\ 10 & 6 & 9 & -13 & 10 & 14 \end{bmatrix}$$

In $H^{*(1)}$ the entries $h^{*(1)}_{2,j}$ and $h^{*(1)}_{5,j}$ are nonzero $\left(i.e.\ h^{*(1)}_{2,2} = 19, h^{*(1)}_{2,3} = 14, h^{*(1)}_{2,4} = -29, h^{*(1)}_{2,5} = 36, h^{*(1)}_{5,2} = 14, h^{*(1)}_{5,3} = \frac{65}{2}, h^{*(1)}_{5,4} = \frac{-97}{2}, h^{*(1)}_{5,5} = 39\right)$ for $j = 2, 3, 4, 5$. Otherwise apply suitable row-interchange in $H^{*(0)}$ and re-factorize, else the factorization breakdown.

Step 3: Now, we can compute the next set of 2×2 systems of linear equation from the entries in $H^{*(1)}$.

Let $k = 2$, then

$$\begin{cases} h^{*(1)}_{2,2} w^{*(2)}_{i,2} + h^{*(1)}_{n-1,2} w^{*(2)}_{i,n-1} = -h^{*(1)}_{i,2} \\ h^{*(1)}_{2,n-1} w^{*(2)}_{i,2} + h^{*(1)}_{n-1,n-1} w^{*(2)}_{i,n-1} = -h^{*(1)}_{i,n-1} \end{cases}$$

Whenever $i = 3$ then

$$\begin{cases} h^{*(1)}_{2,2} w^{*(2)}_{3,2} + h^{*(1)}_{5,2} w^{*(2)}_{3,5} = -h^{*(1)}_{3,2} \\ h^{*(1)}_{2,5} w^{*(2)}_{3,2} + h^{*(1)}_{5,5} w^{*(2)}_{3,5} = -h^{*(1)}_{3,5} \end{cases} \Rightarrow \begin{cases} 19 w^{*(2)}_{3,2} + 14 w^{*(2)}_{3,5} = 36 \\ 36 w^{*(2)}_{3,2} + 39 w^{*(2)}_{3,5} = 42 \end{cases} \Rightarrow \begin{cases} w^{*(2)}_{3,2} = \frac{272}{79} \\ w^{*(2)}_{3,5} = \frac{-166}{79} \end{cases}$$

Whenever $i = 4$ then

$$\begin{cases} h^{*(1)}_{2,2} w^{*(2)}_{4,2} + h^{*(1)}_{5,2} w^{*(2)}_{4,5} = -h^{*(1)}_{4,2} \\ h^{*(1)}_{2,5} w^{*(2)}_{4,2} + h^{*(1)}_{5,5} w^{*(2)}_{4,5} = -h^{*(1)}_{4,5} \end{cases} \Rightarrow \begin{cases} 19 w^{*(2)}_{4,2} + 14 w^{*(2)}_{4,5} = -121 \\ 36 w^{*(2)}_{4,2} + 39 w^{*(2)}_{4,5} = -325 \end{cases} \Rightarrow \begin{cases} w^{*(2)}_{3,2} = \frac{169}{237} \\ w^{*(2)}_{3,5} = \frac{-1819}{237} \end{cases}$$

Thus,

$$W^{*(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -15 & 1 & 0 & 0 & 0 & 4 \\ 30 & \frac{272}{79} & 1 & 0 & \frac{-166}{79} & 0 \\ 176 & \frac{169}{237} & 0 & 1 & \frac{-1819}{237} & \frac{-103}{2} \\ -19 & 0 & 0 & 0 & 1 & \frac{11}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 4: We then proceed to update $H^{*(1)}$ to $H^{*(2)}$ by computing its entries as

$$h^{*(k)}_{i,j} = h^{*(k-1)}_{i,j} + w^{*(k)}_{i,k} h^{*(k-1)}_{k,j} + w^{*(k)}_{i,n-k+1} h^{*(k-1)}_{n-k+1,j} \Rightarrow h^{*(2)}_{i,j} = h^{*(1)}_{i,j} + w^{*(2)}_{i,2} h^{*(1)}_{2,j}$$

When $i = 3$ and $j = 3, 4$ then

$$h^{*(2)}_{3,3} = h^{*(1)}_{3,3} + w^{*(2)}_{3,2} h^{*(1)}_{2,3} + w^{*(2)}_{3,5} h^{*(1)}_{5,3} = -42 + \left(\frac{272}{79}\right)(14) + \left(\frac{-166}{79}\right)\left(\frac{65}{2}\right) = \frac{-9810}{158}$$

$$h^{*(2)}_{3,4} = h^{*(1)}_{3,4} + w^{*(2)}_{3,2} h^{*(1)}_{2,4} + w^{*(2)}_{3,5} h^{*(1)}_{5,4} = 93 + \left(\frac{272}{79}\right)(-29) + \left(\frac{-166}{79}\right)\left(\frac{-97}{2}\right) = \frac{15052}{158}$$

When $i = 4$ and $j = 3, 4$ then

$$h^{*(2)}_{4,3} = h^{*(1)}_{4,3} + w^{*(2)}_{4,2} h^{*(1)}_{2,3} + w^{*(2)}_{4,5} h^{*(1)}_{5,3} = \left(\frac{-559}{2}\right) + \left(\frac{169}{237}\right)(14) + \left(\frac{1819}{237}\right)\left(\frac{65}{2}\right) = \frac{-9511}{474}$$

$$h^{*(2)}_{4,4} = h^{*(1)}_{4,4} + w^{*(2)}_{4,2} h^{*(1)}_{2,4} + w^{*(2)}_{4,5} h^{*(1)}_{5,4} = \left(\frac{947}{2}\right) + \left(\frac{169}{237}\right)(-29) + \left(\frac{1819}{237}\right)\left(\frac{-97}{2}\right) = \frac{38194}{474}$$

Thus,

$$H^{*(2)} = H^{*}_{g} = \begin{bmatrix} 3 & 1 & 1 & -1 & 1 & 4 \\ 0 & 19 & 14 & -29 & 36 & 0 \\ 0 & 0 & \frac{-9810}{158} & \frac{15052}{158} & 0 & 0 \\ 0 & 0 & \frac{-9511}{474} & \frac{38194}{474} & 0 & 0 \\ 0 & 14 & \frac{65}{2} & \frac{-97}{2} & 39 & 0 \\ 10 & 6 & 9 & -13 & 10 & 14 \end{bmatrix}$$

The factorization stops since in $H^{*(2)}$ the entries $h^{*(2)}_{3,j}$ and $h^{*(2)}_{4,j}$ are nonzero, for $j = 3, 4$.

To get the matrix A, we express A as

$$A = \left(W^{*(2)} W^{*(1)} P^{*(1)}\right)^{-1} H^{*(2)}.$$

2.2 Properties of Hourglass Matrix and Its Factorization Algorithm

Though not always, it is important to note that properties of hourglass matrix and Z-matrix are similar. Like Z-matrix, the transpose of hourglass matrix does not retain the shape of the matrix but rather form a bowtie matrix or butterfly matrix. Inverse of hourglass matrix is again hourglass matrix. The minimum order of hourglass matrix is 3 and the matrix cannot be symmetric. The rank of hourglass matrix is n. Like Z-matrix, hourglass matrix is closed under addition and multiplication. More so, hourglass matrix is a general linear group of degree n over $\mathbb{R}$. Regardless of order of hourglass matrix, the total number of zero entries is even. Besides, hourglass matrix has minimum matrix density of 0.5 as $\lim_{n\to\infty} \frac{\frac{n^2+2n-(n+1)\bmod 2-1}{2}}{n^2}$. If $det(f_m^1) = det(f_m^2) = \cdots = det\left(f_m^{\left\lceil\frac{n-1}{2}\right\rceil}\right)$, then it is easy to deduce for the determinant of even order of hourglass matrix as

$$det(H^{*}_{g}) = det\left(f_m^{\frac{n}{2}}\right)^{\frac{n}{2}}.$$

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Irrespective of the order, if the entries in main diagonal of hourglass matrix are 1's then

$$det(H_g^{\bullet}) = \prod_{i=1}^{\lceil \frac{n-1}{2} \rceil} (1 - h_{i,n+i-1}^{\bullet} h_{n+i-1,i}^{\bullet})$$

and $tr(H_g^{\bullet}) = n$.

3.0 NEW RESULTS ON HOURGLASS MATRIX

Since the quadrant interlocking factorization of hourglass matrix has been established, we give the MATLAB code to generate random hourglass matrix of order $n(n \geq 3)$ in Listing 3.1.

Listing 3.1. MATLAB code for random hourglass matrix.

```
function A = random_H(N,k)
A = tril(ones(N));
non_zero_size = nchoosek(N+1,2);
v = zeros(non_zero_size,1);
for i = 1:nchoosek(N+1,2)
    v(i) = gen(k);
    [illegible]
end
A(~~A) = v;
for j = 1:floor(N/2)
    A(j, j+1:N-j) = A(j+1:N-j, j)';
    A(j+1:N-j, j) = zeros(N-2*j, 1);
    A(j, N-j+1) = gen(k);
end
```

Theorem 3.1. If there exists WH factorization for a nonsingular matrix A, then there exists also WZ factorization.

Proof

First, we assume matrix A has even order (the assumption is also true for odd order). If $A = WH$, then the centro-nonsingular submatrices $\nabla_h = h_{i,j}^{\bullet(k-1)}$ of A are nonsingular according to its factorization algorithm, otherwise the factorization fails. That is,

$$\nabla_h = \begin{bmatrix} h_{k,k}^{\bullet(k-1)} & \cdots & h_{k,n+k-1}^{\bullet(k-1)} \\ \vdots & \ddots & \vdots \\ h_{n+k-1,k}^{\bullet(k-1)} & \cdots & h_{n+k-1,n+k-1}^{\bullet(k-1)} \end{bmatrix}$$

This assumption is also applicable to $A = WZ$ according of Factorization theorem (see Rao (1997)), if and only if its centro-nonsingular submatrices $\Delta_z = z_{i,j}^{\bullet(k-1)}$ are invertible. The 2×2 submatrix from centro-nonsingular submatrix has the least condition number adopting any matrix norm, such that

$$\Delta_z = \begin{bmatrix} z_{k,k}^{\bullet(k-1)} & \cdots & z_{k,n+k-1}^{\bullet(k-1)} \\ \vdots & \ddots & \vdots \\ z_{n+k-1,k}^{\bullet(k-1)} & \cdots & z_{n+k-1,n+k-1}^{\bullet(k-1)} \end{bmatrix}$$

If a nonsingular matrix A with centro-nonsingular submatrix assumes WH factorization such that

$$\det(\nabla_h) = h_{k,k}^{\bullet(k-1)} h_{n+k-1,n+k-1}^{\bullet(k-1)} - h_{k,n+k-1}^{\bullet(k-1)} h_{n+k-1,k}^{\bullet(k-1)} \neq 0$$

then the matrix also assumes WZ factorization such that

$$\det(\Delta_z) = z_{k,k}^{\bullet(k-1)} z_{n+k-1,n+k-1}^{\bullet(k-1)} - z_{k,n+k-1}^{\bullet(k-1)} z_{n+k-1,k}^{\bullet(k-1)} \neq 0$$

However, the computed entry $z_{i,j}^{\bullet(k-1)}$ may or may not be nonzero for $i,j = k, k+1, \ldots, n-k+1$. This is because WZ factorization only requires invertibility of Δ_z, whereas WH factorization ensures that row interchange exists for ∇_h to contain only nonzero entries and still being invertible. In a case where $z_{i,j}^{\bullet(k-1)} \neq 0$, then $z_{i,j}^{\bullet(k-1)} = h_{i,j}^{\bullet(k-1)}$ but if $z_{i,j}^{\bullet(k-1)} = 0$, then $z_{i,j}^{\bullet(k-1)} \neq h_{i,j}^{\bullet(k-1)}$ even though $\Delta_z \neq 0$ and $\nabla_h \neq 0$. Thus, WH factorization does not always imply WZ factorization.

Corollary 3.1. Every hourglass matrix is invertible.

Proof

Based on Factorization Theorem for WZ factorization, WH factorization exists because the subcentral matrices ∇_k of $H_g^{\bullet}$ must not only be nonzero but also nonsingular. Hence, it can be infered from Theorem 3.1 that H_g^* is invertible else the factorization will not exist to yield hourglass matrix.

4.0 DISCUSSION OF RESULTS

Although, H-matrix and Z-matrix (especially when factorized from Hankel and Toeplitz matrix) share most things in common even with X-matrix, yet Z-matrix does not always imply hourglass matrix because Z-matrix is more general than hourglass matrix. The quadrant interlocking factorization of Z-matrix is possible provided the submatrices of the nonsingular matrix are invertible. Whereas, quadrant interlocking factorization of hourglass matrix is possible provided the submatrices of the nonsingular matrix are invertible as well as all the elements in the first row and in the last row of each submatrix are nonzero. Assuming the entries $h_{i,j}^*$ is analogous to $z_{i,j}$, then Z-matrix will imply hourglass matrix (of even order) provided that the computed $z_{1,j}^{(k-1)}$ and $z_{n,j}^{(k-1)}$ are strictly nonzero, for $k = 1, 2, \ldots, \frac{n}{2}$. However, the entries of Z-matrix are unbound to be nonzero. Therefore, quadrant interlocking factorization of symmetric positive definite or diagonally dominant does not guarantee that the factored matrix is hourglass matrix, however it guarantees that the factored matrix is Z-matrix. Then it is obvious that it will no longer be an hourglass matrix if one of its strictly nonzero elements is replaced with zero. If the main diagonal entries of hourglass matrix are 1's and the anti-diagonal entries are replaced with 0's, then it is referred to as unit Z-matrix and it is called unit split Z-matrix if the anti-diagonal entries are replaced with 1's. In general, every H-matrix is a Z-matrix, but the converse is not true. That is, H-matrix is a subset of Z-matrix. The total number of zero entries in hourglass matrix is even irrespective of its order. For every odd order of hourglass matrix, the epicenter element does not belong to filanz matrix. Lastly, unlike Z-matrix, hourglass matrix can be represented as mixed graph.

5.0 CONCLUSION

The notion of quadrant interlocking factorization of hourglass matrix (H-matrix) was successfully discussed with fascinating results on its properties and its differences from Z-matrix. WZ factorization is more general than WH factorization because, a matrix that factorizes into Z-matrix may not able to factorize into H-matrix. Thus, H-matrix is a subset of Z-matrix. Further research may reveal how this matrix can be used in quantum information theory especially entanglement.

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