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TROPICALIZATION OF IDEMPOTENT ELEMENTS ON FULL TRANSFORMATION SEMIGROUP

Usamot, I. F., Bakare, G. N. & Ahmed, B. M.

Department of Mathematics,
University of Ilorin
Email: usamot.if@unilorin.edu.ng

ABSTRACT

This paper consists of exposition for tropical geometry and its applications on Idempotent elements in Semigroup of Full Transformation $E(T_n)$. We started the exposition with classical algebra into tropical algebra and shown how tropical curves degenerate tropical polynomial in $E(T_n)$.

Keywords: Tropical polynomial, Tropical curve, Semigroup, Idempotent, Trace, Full transformation, Height, Multiplicity.

1.0 INTRODUCTION

Tropical geometry has established itself as an important new field bridging algebraic geometry whose techniques have been used to attack problems, these include enumerative geometry and arithmetic geometry. It builds on the older area of tropical mathematics more commonly known as max-plus algebra which arises in semigroup theory, computer science and optimization. The name "tropical" was coined by French mathematician which is denoted by T . As in classical algebra, a tropical polynomial expression $P(x) = "\sum_{i=0}^{d} a_i x^i"$ induces a tropical polynomial function denoted by P on T ;

$$P : \mathrm{T} \to \mathrm{T} \qquad (1)$$

A strange space with mysterious properties hide behind the enigmatic name of tropical geometry. This strange object is a line because of the geometric properties it satisfied. The set of tropical numbers is defined as $\mathrm{T} = \mathrm{R} \cup \{-\infty\}$ endowed with the operations called tropical addition and multiplication:

$$"x + y" = max\{x, y\}; \qquad (2)$$

$$"x \times y" = x + y \qquad (3)$$

with the usual conventions:

$$\forall x \in , "x + (-\infty)" = max(x, -\infty) = x$$
$$and \qquad (4)$$
$$"x \times (-\infty)" = x + (-\infty) = -\infty$$

The tropical numbers along with these two operations form a semi-field, that is, they satisfies all the axioms of a field except the existence of an inverse for the law of addition. In tropical operations, $"2x" \neq "x + x"$ but $"2x" = x + 2$ and $"0x" = x$ but not equal to 0. In this paper, tropical operations will be placed under quotation marks. A tropical polynomial is a convex piecewise affine function and each piece has an integer slope.

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation while a transformation of X is a function from X to itself. Let $X_n = \{1, 2, 3, ..., n\}$ be a finite set ordered in

*Corresponding Author

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standard way. A map $a: Dom(a) \subseteq X_n \to Im(a) \subseteq X_n$ is called full or total transformation of X_n if $Dom(a) = X_n$ denoted by T_n. The set T_n of a full transformations on X_n forms a semigroup under composition of mappings called the full transformation semigroup and the trace of matrix A denoted by $tr(A)$ defined as the sum of the diagonal matrix A.

For standard concepts and terms in semigroup and tropical geometry theory see [Howie(1995); Umar (2010); Itenberg et. al. (2007), Brugalle (2012); Brugalle and Shaw (2014); Maclagan (2012)] and so on.

Some researchers have worked extensively on tropical geometry. Recently, Katz (2012) enumirated the lifting tropical curves in space and linear systems on graph and Shaw (2013) also studied a tropical intersection product in matroidal fans.

The goal of this paper is to introduce tropical geometry on semigroups of full transformation T_n. We begin by degenerating some polynomial functions of T_n into tropical polynomials and established some theorems with examples to validate results.

Definition (Brugalle et. al. 2015)

A tropical number r is a root of a Polynomial $P(x)$ in one variable in which the points x_0 on the graph $P(x)$ has a corner at x_0 for $-\infty \le r \le \infty$.

Definition (Brugalle et. al. 2015)

The Multiplicity of a tropical root r denoted by $M(r)$ defined as

$M(r) = |m_i - m_{i-1}|$

where m_i are the slopes of the lines in the tropical graph $P(x)$ intersecting above r.

Proposition (Brugalle et. al. 2015)

The tropical semi-field is algebraically closed. That is, every tropical polynomial of degree $d > 0$ has exactly d roots when counted with multiplicities.

Definition (Umar 2010)

Let $S = T_n$ and $a \subset T_n$. Then, the height of $a = |Im a|$

Definition (Umar 2010)

An element e in T_n is idempotent if $e^2 = e$ and if and only if $|Im e| = |F(e)|$.

2.0 MAIN RESULTS

Lemma 1: Let $S = T_n$ and $E(T_n)$ be the set of idempotent elements in full transformation semigroup then, for any $n \times n$ matrix of $a, b \in E(T_n)$;

$(i) tr(a) = tr(b)$ for $a_{11} < a_{22} ..... < a_{nn}$ and $|a| \neq |b|$

$(ii) a = a^T$.

Theorem 2: Every $E(T_n)$ in tropical has multiplicities of the same height.

Proof: Let $S = T_n$ be a semigroup of full transformation on X_n and $E(T_n)$ be the idempotent elements in T_n. Suppose $a, b \subset E(T_n)$ and $a(x), b(x), g(x)$ be their polynomials respectively. Also, let $M\Gamma_{a(x)}, M\Gamma_{b(x)}, M\Gamma_{g(x)}$ be the multiplicity of tropical polynomials a, b and g with the slopes m_i for

$(i = 1,2,\ldots,k)$.

If $a = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $b = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$ and $g = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$ with polynomial functions;

$$a(x) = a_{11}x^4 + (a_{12} + a_{21})x^3 + (a_{13} + a_{22} + a_{31})x^2 + (a_{23} + a_{32})x + a_{33};$$

$$b(x) = b_{11}x^4 + (b_{12} + b_{21})x^3 + (b_{13} + b_{22} + b_{31})x^2 + (b_{23} + b_{32})x + b_{33}$$

and

$$g(x) = c_{11}x^4 + (c_{12} + c_{21})x^3 + (c_{13} + c_{22} + c_{31})x^2 + (c_{23} + c_{32})x + c_{33}$$

for all $a_{ij} \neq b_{ij} \neq c_{ij}$.

Consequently, by virtue of (2) and (3) we have;

$$"a_{11}x^4 + (a_{12} + a_{21})x^3 + (a_{13} + a_{22} + a_{31})x^2 + (a_{23} + a_{32})x + a_{33}"$$
$$= max(a_{11} + 4x, a_{12} + a_{21} + 3x, a_{13} + a_{22} + a_{31} + 2x, a_{23} + a_{32} + x, a_{33});$$

$$"b_{11}x^4 + (b_{12} + b_{21})x^3 + (b_{13} + b_{22} + b_{31})x^2 + (b_{23} + b_{32})x + b_{33}"$$
$$= max(b_{11} + 4x, b_{12} + b_{21} + 3x, b_{13} + b_{22} + b_{31} + 2x, b_{23} + b_{32} + x, b_{33})$$

and

$$"c_{11}x^4 + (c_{12} + c_{21})x^3 + (c_{13} + c_{22} + c_{31})x^2 + (c_{23} + c_{32})x + c_{33}"$$
$$= max(c_{11} + 4x, c_{12} + c_{21} + 3x, c_{13} + c_{22} + c_{31} + 2x, c_{23} + c_{32} + x, c_{33})$$

Observe that, the tropical polynomials have the same order of roots with different values and slopes m_i emanated from the tropical graph. Then, we deduced that the multiplicities of these tropical polynomials through the tropical curves have the same height; that is

$|M(\mathrm{T}_{a(x)})| = |M(\mathrm{T}_{b(x)})| = |M(\mathrm{T}_{g(x)})|$.

Hence the proof.

Example 1: Consider

$a, b, g \subseteq E(T_3)$ $a = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix}$,

$b = \begin{pmatrix} 1 & 3 & 3 \\ 3 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}$ and

$g = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$

with polynomial functions;

$$a(x) = x^4 + 2x^3 + x^2 + x^3 + 2x^2 + 2x + x^2 + x + 3$$
$$= x^4 + 3x^3 + 4x^2 + 3x + 3;$$

$$b(x) = x^4 + 3x^3 + 3x^2 + 3x^3 + 2x^2 + 3x + 2x^2 + 2x + 3$$
$$= x^4 + 6x^3 + 7x^2 + 5x + 3$$

and

$$g(x) = x^4 + x^3 + x^2 + 2x^3 + 2x^2 + 2x + 3x^2 + 3x + 3$$

$$= x^4 + 3x^3 + 6x^2 + 5x + 3$$

Then, by the virtue of (2) and (3) we have;

$$"x^4 + 3x^3 + 4x^2 + 3x + 3" = max(4x, 3x + 3, 2x + 4, x + 3, 3)$$
;

$$"x^4 + 6x^3 + 7x^2 + 5x + 3" = max(4x, 3x + 6, 2x + 7, x + 5, 3)$$

and

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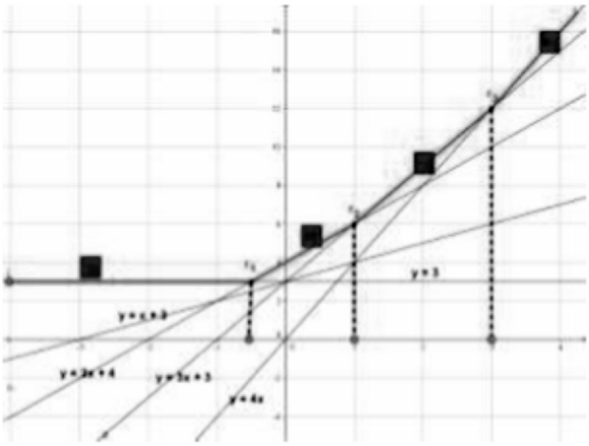
$$"x^4+3x^3+6x^2+5x+3"=max(4x,3x+3,2x+6,x+5,3)$$

Tropical curve of $a(x)$



Graph I

From the above tropical curve, the roots are; $r_1=-0.5, r_2=1$ and $r_3=3$ with slopes $m_1=0, m_2=2, m_3=3$ and $m_4=4$ having multiplicities

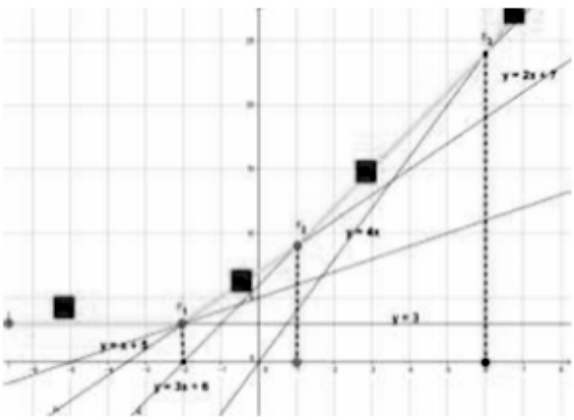
$$M(r_1)=|m_1-m_2| \\ =2$$

$$M(r_2)=|m_2-m_3| \\ =1$$

$$M(r_3)=|m_3-m_4| \\ =1$$

Hence, the multiplicities of $\mathrm{T}_{a(x)}=(2\ 1\ 1)$.

Tropical curve of $b(x)$



Graph II

Similarly, the roots of $\mathrm{T}_{b(x)}$ is given as $r_1=-2, r_2=1$ and $r_3=6$ having slopes $m_1=0, m_2=2, m_3=3$ and $m_4=4$ with multiplicities

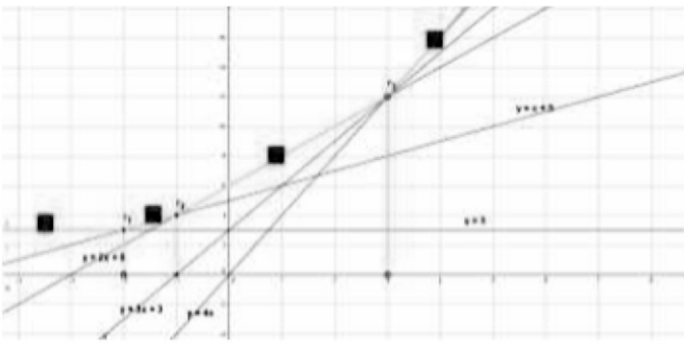
$$M(r_1)=|m_1-m_2| \\ =2$$

$$M(r_2)=|m_2-m_3| \\ =1$$

$$M(r_3)=|m_3-m_4| \\ =1$$

Thus, the multiplicities of $\mathrm{T}_{b(x)}=(2\ 1\ 1)$

Tropical curve of $g(x)$



Graph III.

Moreso from Graph III, the roots of $\mathrm{T}_{g(x)}$ were;

$r_1=-2, r_2=-1, r_3=3$ having slopes $m_1=0, m_2=1, m_3=2$ and $m_4=4$ with multiplicities;

$$M(r_1)=|m_1-m_2| \\ =1\ ,$$

$$M(r_2)=|m_2-m_3| \\ =1\ ,$$

$$M(r_3)=|m_3-m_4| \\ =2\ .$$

Then, the multiplicities of $\mathrm{T}_{g(x)}=(1\ 1\ 2)$

Hence,

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$\left|M(\mathrm{T}_{a(x)})\right| = \left|M(\mathrm{T}_{b(x)})\right| = \left|M(\mathrm{T}_{g(x)})\right|$.

Example 2: Consider Q and R which are the set of ranges in $E(T_4)$.

Suppose $Q = \begin{pmatrix} 1 & 3 & 3 & 3 \\ 2 & 2 & 3 & 4 \end{pmatrix}$ and

$R = \begin{pmatrix} 2 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 \end{pmatrix}$

with polynomial functions;

$$Q(x) = x^4 + 3x^3 + 3x^2 + 3x + 2x^3 + 2x^2 + 3x + 4$$

$$= x^4 + 5x^3 + 5x^2 + 6x + 4$$

and

$$R(x) = 2x^4 + 2x^3 + 3x^2 + 4x + 2x^3 + 2x^2 + 3x + 3$$

$$= 2x^4 + 4x^3 + 5x^2 + 7x + 3$$

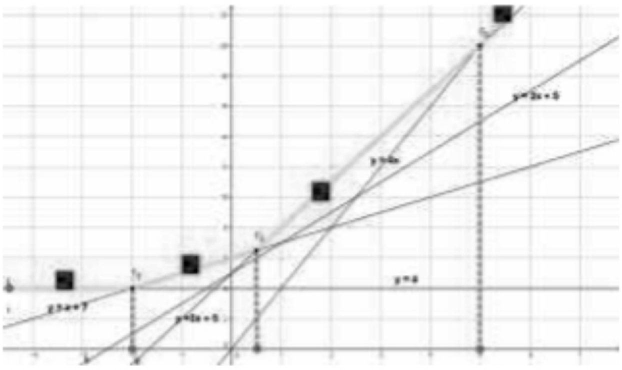
Again, by (2) and (3) we have;

$$"x^4+5x^3+5x^2+6x+4" = max(x+4, 3x+5, 2x+5, x+6, 4)$$

and

$$"2x^4+4x^3+5x^2+7x+3" = max(2x+4, 3x+4, 2x+5, x+7, 3)$$

Tropical curve of $Q(x)$



Graph IV.

Then, the GraphIV of $\mathrm{T}_{Q(x)}$ have roots; $r_1 = -2, r_2 = 0.5, r_3 = 5$ with slopes $m_1 = 0, m_2 = 1, m_3 = 3$ and $m_4 = 4$.

So, the multiplicity of these roots were;

$$M(r_1) = |m_1 - m_2|$$

$$= 1 \quad ,$$

$$M(r_2) = |m_2 - m_3|$$

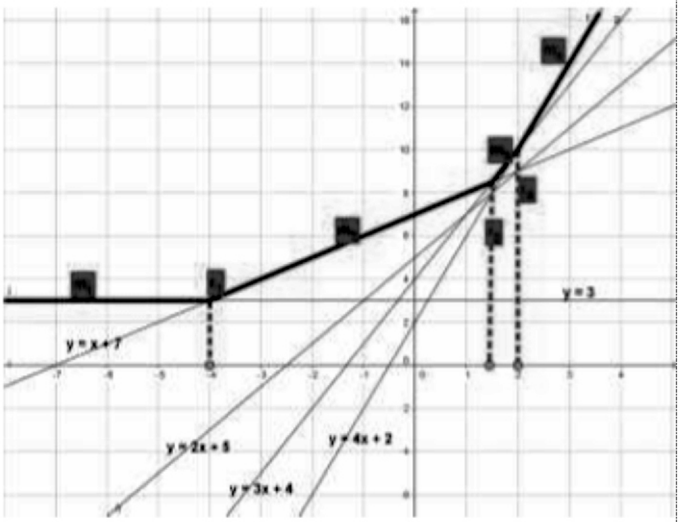
$$= 2 \quad ,$$

$$M(r_3) = |m_3 - m_4|$$

$$= 1 \quad .$$

Therefore, the multiplicities of $\mathrm{T}_{Q(x)} = (1 \;\; 2 \;\; 1)$

Tropical curve of $R(x)$



Graph V.

Similarly from graph V, the roots of $\mathrm{T}_{R(x)}$ were given as follows;

$r_1 = -4, r_2 = 1.5, r_3 = 2$ with slopes $m_1 = 0, m_2 = 1, m_3 = 3$, and $m_4 = 4$. Then,

$$M(r_1) = |m_1 - m_2|$$

$$= 1 \quad ,$$

$$M(r_2) = |m_2 - m_3|$$

$$= 2$$

and

$$M(r_3) = |m_3 - m_4|$$

$$= 1$$

having the multiplicities $(1 \;\; 2 \;\; 1)$.

Hence, $\left|M(\mathrm{T}_{Q(x)})\right| = \left|M(\mathrm{T}_{R(x)})\right|$

Theorem 3: Let $S = T_n$ and $A \subset E(T_n)$. Also, let T be a tropical and $A(x)$ be a polynomial function in $E(T_n)$. Then, $A(x) = A^T(x) \Rightarrow \mathrm{T}_{A(x)} = \mathrm{T}_{A^T(x)}$.

Proof:

Suppose $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ and

$$A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

Then,

$$A(x) = a_{11}x^3 + (a_{12} + a_{21})x^2 + (a_{13} + a_{22})x + a_{23}$$

$$A^T(x) = a_{11}x^3 + (a_{21} + a_{12})x^2 + (a_{22} + a_{13})x + a_{23}$$

By the virtue of (2), (3) and condition (ii) of Lemma (1) we have;

$$"a_{11}x^3 + (a_{12} + a_{21})x^2 + (a_{13} + a_{22})x + a_{23}" = max(a_{11} + 3x, a_{12} + a_{21} + 2x, a_{13} + a_{22} + x, a_{23})$$

and

$$"a_{11}x^3 + (a_{21} + a_{12})x^2 + (a_{22} + a_{13})x + a_{23}" = max(a_{11} + 3x, a_{21} + a_{12} + 2x, a_{22} + a_{13} + x, a_{23})$$

implies that,

$$A(x) = A^T(x) \text{ and } \mathrm{T}_{A(x)} = \mathrm{T}_{A^T(x)}.$$

Hence, the proof.

3.0 CONCLUSION.

In this paper, we have shown that in semigroup of full transformations, every idempotent elements of different tropical polynomials have multiplicities of the same height despite their differences in tropical roots and slopes which were obtained from the tropical graph through GeoGebra. More so, every polynomial functions on E(Tn) and its transpose have the same tropical polynomials

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