# DEVELOPMENT OF ADAMS BASHFORTH METHODS USING CHEBYSHEV POLYNOMIAL AS BASIS FUNCTION

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**ABSTRACT**

The important fundamental properties of numerical methods for ordinary differential equations are investigated. This involves the derivation of Adam Bashforth method for k=2, 3, 4 and k=5, together with their continuous forms using Chebyshev polynomials as basis function. The formulas derived were also used to solve Initial Valued Problems (IVP). The results agree remarkably with those from the literature.

***Keywords:*** you must provide keywords here

## 1. INTRODUCTION

The method for the numerical solution of initial value problem (IVP) is as important as the solution itself. There are several methods adopted for the numerical solution of IVP, Examples are: Adam-Moulton, Runge-Kutta, Euler's rule etc. They all have their inherent advantages and disadvantages – The Euler's rule is known as explicit, one step method and being a one-step method, it requires no additional starting values. Its low order makes it of limited practical values.

Linear Multi-Step Methods(LMMs) on the other hand are not self-starting hence, need starting values from single-step methods like Euler's method.

The general k- step method or LMM of step number k given by Lambert (1973) is written as:

$$\sum_{j=0}^{k} a_j y_{n+j} = h\sum_{j=0}^{k} b_j(x) f_{n+j}\ , a_k \neq 0 \qquad (1.1)$$

Where $a_j and b_j$ are uniquely determined and h is the step length such that $x_{k+n} - x_k = nh$

Linear Multi-Step LMMs achieve higher order by sacrificing the One-Step nature while retaining linearity with respect to $y_{n-j,} f_{n+j,} j = 0,1,\dots k$.

Higher order can also be achieved by sacrificing linearity but preserving the one-step nature. This is the rationale behind the method proposed by Runge and subsequently developed by Kutta and Heun.

The LMM generate discrete multistep schemes (1.1) which are used for solving the Initial Valued Problem (IVP):

$$y'(x) = f(x, y(x)),\ y(x_0) = y_0 \qquad (1.2)$$

In Awoyemi(1999) and Onumanyi et al(1993), some continuous LMMs of the type in Equation (1.1) were developed using power series collocation function of the form:

$$y(x) = \sum_{j=0}^{k} a_j x^j \qquad (1.3)$$

Okunuga and Ehigie (2009) used similar function of the form:

$$y(x) = \sum_{j=0}^{k} a_j (x - x_k)^j \qquad (1.4)$$

However, Adeniyi and Alabi (2006) used Chebyshev polynomial function of the form

$$y(x) = \sum_{j=0}^{m} a_j T_j(x)\left(\frac{x - x_k}{h}\right) \qquad (1.5)$$

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where $T_j(x)$ are some Chebyshev function to develop continuous LMM.

These shifted polynomials will not be used in the present work, as we would like to take advantage of the many symmetry properties of the ordinary Chebyshev polynomials about x =0.

Thus, in this paper, we shall use the unmodified Chebyshev polynomial function as basis function for the derivation of Euler's method and Adam Bashforth methods for $k = 2, k = 3, k = 4, k = 5$. That is, we shall assume a solution of the IVP (1.2) to be of this form:

$$y(x) = \sum_{j=0}^{t} a_j T_j(x) \tag{1.6}$$

## 2. CHEBYSHEV POLYNOMIALS

In this section, we give an introduction to the Chebyshev polynomials and their basic properties. See the references (Fox and Parker,1968, William,1974, Theodore,1990, Reutskiy and Chen, 2006) for more details.

The Chebyshev polynomials of the first kind (Richard and Douglas, 2001) arise as the solution to the Chebyshev differential equation

$$(1-x^2)y''(x) - xy'(x) + N^2 y(x) = 0. \tag{2.1}$$

There are defined by the recurrent relation

$$T_0(x) = 1, \tag{2.2}$$

$$T_1(x) = x, \tag{2.3}$$

$$T_{N+1}(x) = 2xT_N(x) - T_{N-1}(x). \tag{2.4}$$

They can alternatively be defined by the trigonometric identity:

$$T_N(x) = \cos(N\cos^{-1} x), \tag{2.5}$$

where $T_N(\cos(q)) = \cos(Nq),\ for\ N = 0,1,2,...$

The Chebyshev polynomials of the first kind are orthogonal with respect to the weight function

$$w(x) = \frac{1}{\sqrt{1-x^2}} \text{ on the interval } [-1,1], \text{ that is}$$

$$\int_{-1}^{1} \frac{T_N(x)T_M}{\sqrt{1-x^2}}dx = \begin{cases} 0 & N \neq M, \\ p & N = M = 0, \\ \frac{p}{2} & N = M \neq 0. \end{cases} \tag{2.6}$$

One can easily show that the roots of

$$T_N(x) \text{ are } x_k = \cos\left(\frac{(2k-1)p}{2N}\right),\ k = 1,2,...,N. \tag{2.7}$$

The Chebyshev polynomials of the second kind (Martin, 1966) arise as the solution to the Chebyshev differential equation

Vol. 2, No. 1, June 2018
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$$(1-x^2)y''(x)-3xy'+N(N+2)y(x)=0. \quad (2.8)$$

They are defined by the recurrence relation

$$U_0(x)=1, \quad (2.9)$$

$$U_1(x)=2x, \quad (2.10)$$

$$U_{N+1}(x)=2xU_N(x)-U_{N-1}(x). \quad (2.11)$$

They can alternatively be defined by the trigonometric identity:

$$U_N(x)=\frac{\sin((N+1)q)}{\sin q}, \quad (2.12)$$

with $x=\cos q$

They are orthogonal with respect to the weight function $w(x)=\sqrt{1-x^2}$ on the interval $[-1, 1]$, that is

$$\int_{-1}^{1} U_N(x)U_M(x)\sqrt{1-x^2}\,dx=\begin{cases}0 & N\neq M,\\ \frac{p}{2} & N=M.\end{cases} \quad (2.13)$$

The roots of $U_N(x)$ are given by

$$x_k=\cos\left(\frac{kp}{N+1}\right), \quad k=1, 2, \ldots, N. \quad (2.14)$$

The Chebshev polynomials of the first and second kind are closely related. For example, a Chebyshev polynomial of first kind can be represented as a linear combination of two Chebyshev polynomials of second kind,

$$T_N(x)=\frac{1}{2}(U_N(x)-U_{N-2}(x)),$$

and the derivative of a Chebyshev polynomial of first kind can be written in terms of a Chebyshev polynomial of second kind,

$$T_N'(x)=NU_{N-1}(x), \ N=1, 2, \ldots$$

Thus in this paper, we focus our attentation to the Chebyshev polynomials of first kind and we use them for approximating a function and a particular solution for IVPs.

The first few Chebyshev polynomials are listed below:

$$T_0(x)=1,$$
$$T_1(x)=x,$$
$$T_2(x)=2x^2-1,$$
$$T_3(x)=4x^3-3x,$$
$$T_4(x)=8x^4-8x^2+1,$$
$$T_5(x)=16x^5-20x^3+5x,$$
$$T_6(x)=32x^6-48x^4+18x^2-1,$$
$$\cdots$$

## 3 DERIVATION OF EULER'S METHOD

We consider the equation

$$y'(x)=f(x,y(x)),\ y(x_0)=y_0$$

We assume a solution of the form:

$$y(x)=\sum_{j=0}^{k} a_jT_j(x) \quad (3.1)$$

such that $T_j(x)$ is the basis function, $y(x)$ is a dependent function

When $k=1$, from equation (3.1) we have

$$y(x)=\sum_{j=0}^{1}a_jT_j(x)=\ a_0T_0(x)+a_1T_1(x) \qquad (3.2)$$

Since $T_0(x)=1$ and $T_1(x)=x$, we have

$$y(x)=a_0+a_1x \qquad (3.3)$$

$$y(x)=f(x,y(x))=a_1$$

At $x=x_n, y'(x_n)=f(x_n,y(x_n))=f_n$

$$f_n=a_1 \qquad (3.4)$$

From (3.3), we have

$$y(x_n)=a_0+a_1x_n$$

That is,

$$y_n=a_0+a_1x_n$$

$$\Rightarrow\ a_0=y_n-f_nx_n \qquad (3.5)$$

Substituting (3.4) and (3.5) into (3.3), we obtain

$$y(x)=y_n-f_nx_n+f_nx \qquad (3.6)$$

Evaluating (3.6) at $x_{n+1}$ gives $y(x_{n+1})=y_n+f_n(x_{n+1}-x_n)$

That is

$$y_{n-1}=y_n+hf_n \qquad (3.7)$$

which is the Euler's rule?

## 4. DERIVATION OF ADAM BASHFORTH METHOD WHEN $k=2$

The Linear Multistep Methods (LMM) for solving the IVP (1.1) can similarly be developed using Chebyshev polynomials of the first kind. The general k- step method or LMM of step number k given by Lambert(1973) is written as:

$$\sum_{j=0}^{k}a_jy_{n+j}=h\sum_{j=0}^{k}b_j(x)f_{n+j}\ ,a_k\neq 0 \qquad (4.1)$$

where $a_j$ and $b_j$ are uniquely determined. h is the step length such that , $x_{k+n}-x_k=nh$

The Linear Multistep Methods (LMM) generates discrete multistep schemes which are used for solving the IVP (1.1). As usual assume a general solution of the form

$$y(x)=\sum_{j=0}^{k}a_jT_j(x)\quad . \qquad (4.3)$$

If k=2 then

$$y(x)=\sum_{j=0}^{2}a_jT_j(x)$$ where $T_j(x)$ is the usual Chebeshev polynomials.

$$y(x)=a_0T_0(x)+a_1T_1(x)+a_2T_2(x)$$

$$y(x)=a_0.1+a_1x+a_2(2x^2-1)$$

Vol. 2, No. 1, June 2018
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$$y(x) = a_0 + a_1 x + a_2(2x^2 - 1)$$

(4.4) At $x = x_{n+1}$, $y(x_{n+1}) = a_0 + a_1(x_{n+1}) + a_2(2x_{n+1}^2 - 1)$

$$y'(x) = a_0 T_0'(x) + a_1 T_1'(x) + a_2 T_2'(x)$$

$$y'(x) = a_1 + a_2 4x \qquad (4.5)$$

At $x = x_n$, $y'(x_n) = f_n = a_1 + 4x_n a_2$

At $x = x_{n+1}$, $y'(x_{n+1}) = f_{n+1} = a_1 + 4x_{n+1} a_2$

$$\Rightarrow y_{n+1} = a_0 + a_1 x_{n+1} + a_2(2x_{n+1}^2 - 1) \qquad (4.6)$$

$$f_n = a_1 + 4x_n a_2 \qquad (4.7)$$

$$f_{n+1} = a_1 + 4a_2 x_{n+1} \qquad (4.8)$$

In matrix form we have

$$\begin{pmatrix} 1 & x_{n+1} & 2x_{n+1}^2 - 1 \\ 0 & 1 & 4x_n \\ 0 & 1 & 4x_{n+1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_{n+1} \\ f_n \\ f_{n+1} \end{pmatrix}$$

Recall that $x_{n+1} = x_n + h$ and letting $[x_n = 0]$

$$h = x_{n+1} - x_n \Rightarrow h = x_{n+1}$$

$$\Rightarrow 2x_{n+1}^2 - 1 = 2h^2 - 1$$

$$\begin{pmatrix} 1 & h & 2h^2 - 1 \\ 0 & 1 & 0 \\ 0 & 1 & 4h \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_{n+1} \\ f_n \\ f_{n+1} \end{pmatrix}$$

Using Cramer's Rule; the determinant of

$$A = \begin{pmatrix} 1 & h & 2h^2 - 1 \\ 0 & 1 & 0 \\ 0 & 1 & 4h \end{pmatrix}$$

is

$$|A| = 1(4h - 0) - h(0) + (2h^2 - 1)(0) = 4h \qquad (4.9)$$

$$A_0 = \begin{pmatrix} y_{n+1} & h & 2h^2 - 1 \\ f_n & 1 & 0 \\ f_{n+1} & 1 & 4h \end{pmatrix}$$

$$|A_0| = y_{n+1}(4h) - h(4hf_n) + (2h^2 - 1)(f_n - f_{n+1}) \qquad (4.10)$$

$$= 4hy_{n+1} - 2h^2 f_n - 2h^2 f_{n+1} - f_n + f_{n+1}$$

$$= 4hy_{n+1} - 2h^2 f_n - 2h^2 f_{n+1} - f_n + f_{n+1}$$

$$A_1 = \begin{pmatrix} 1 & y_{n+1} & 2h^2 - 1 \\ 0 & f_n & 0 \\ 0 & f_{n+1} & 4h \end{pmatrix}$$

$$|A_1| = 1(4hf_n) - y_{n+1}(0) + (2h^2 - 1)(0) = 4hf_n \qquad (4.11)$$

$$A_2 = \begin{pmatrix} 1 & h & y_{n+1} \\ 0 & 1 & f_n \\ 0 & 1 & f_{n+1} \end{pmatrix}$$

$$|A_2| = 1(f_{n+1} - f_n) - h(0) + y_{n+1}(0) \qquad (4.12)$$

$$= f_{n+1} - f_n$$

$$\Rightarrow a_0 = \frac{|A_0|}{|A|} = \frac{4hy_{n+1} - 2h^2 f_n - 2h^2 f_{n+1} - f_n + f_{n+1}}{4h}$$

$$a_1 = \frac{|A_1|}{|A|} = \frac{4hf_n}{4h} = f_n$$

$$a_2 = \frac{|A_2|}{|A|} = \frac{f_{n+1} - f_n}{4h} = \frac{f_{n+1}}{4h} - \frac{f_n}{4h}$$

**From (4.4), we have**

$$y(x) = a_0 + a_1 x + a_2(2x^2 - 1),$$

$$y(x) = \frac{4hy_{n+1} - 2h^2 f_n - 2h^2 f_{n+1} - f_n + f_{n+1}}{4h} + \frac{f_n x}{1} + \frac{(f_{n+1} - f_n)(2x^2 - 1)}{4h}$$

$$y(th) = \frac{4hy_{n+1} - 2h^2 f_n - 2h^2 f_{n+1} - f_n + f_{n+1} + 4hxf_n + 2x^2 f_{n+1} - 2x^2 f_n - f_{n+1} + f_n}{4h}$$

At $x = 2h$

$$y(x_{n+2}) = \frac{4hy_{n+1} - 2h^2 f_n - 2h^2 f_{n+1} - f_n + f_{n+1} + 8h^2 f_n + 8h^2 f_{n+1} - f_{n+1} - 8h^2 f_n + f_n}{4h}$$

$$= y(x_{n+2}) = \frac{4hy_{n+1} - 2h^2 f_n + 6h^2 f_{n+1}}{4h}$$

$$y_{n+2} = y_{n+1} - \frac{1}{2}hf_n + \frac{3}{2}hf_{n+1}$$

$$\Rightarrow y_{n+2} = y_{n+1} - \frac{h}{2}[3f_{n+1} - f_n]$$

## 5. DERIVATION OF ADAMS BASHFORTH METHOD, WHEN $k = 3$

$$y(x) = \sum_{j=0}^{3} a_j T_j(x) \qquad (5.2)$$

$$y(x) = a_0 T_0(x) + a_1 T_1(x) + a_2 T_2(x) + a_3 T_3(x)$$

$$y(x) = a_0.1 + a_1 x + a_2(2x^2 - 1) + a_3(4x^3 - 3x) \qquad (5.3)$$

Vol. 2, No. 1, June 2018
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When $x = x_{n+2},$ we have

$$y(x_{n+2}) = a_0.1 + a_1x_{n+2} + a_2(2x_{n+2}^2 - 1) + a_3(4x_{n+2}^3 - 3x_{n+2})$$

When $x = x_{n+1}$ in (5.3), we have

$$y(x_{n+1}) = a_0 + a_1x_{n+1} + a_2(2x_{n+1}^2 - 1) + a_3(4x_{n+1}^3 - 3x_{n+1})$$

$$y'(x) = a_0T_0'(x) + a_1T_1'(x) + a_2T_2'(x) + a_3T_3'(x)$$

$$y'(x_n) = f_n = a_1(x_n) + a_2 4x_n + a_3(12x_n^2 - 3) \quad (5.5)$$

$$y'(x_{n+1}) = f_{n+1} = a_1(x_{n+1}) + a_2 4x_{n+1} + a_3(12x_{n+1}^2 - 3) \quad (5.6)$$

$$y'(x_{n+2}) = f_{n+2} = a_1(x_{n+2}) + a_2 4x_{n+2} + a_3(12x_{n+2}^2 - 3) \quad (5.7)$$

$$y_{n+2} = a_0 + a_1(x_{n+2}) + a_2(2x_{n+2}^2 - 1) + a_3(4x_{n+2}^3 - 3x_{n+2}) \quad (5.8)$$

$$f_n = 0 + a_1 + a_2 4x_n + a_3(12x_n^2 - 3) \quad (5.9)$$

$$f_{n+1} = 0 + a_1 + a_2 4x_{n+1} + a_3(12x_{n+1}^2 - 3) \quad (5.10)$$

$$f_{n+2} = 0 + a_1 + a_2 4x_{n+2} + a_3(12x_{n+2}^2 - 3) \quad (5.11)$$

$$\Rightarrow Ax = B$$

$$\begin{pmatrix} 1 & x_{n+2} & 2x_{n+2}^2 - 1 4x_{n+2}^3 - 3x_{n+2} \\ 0 & 1 & 4x_n & 12x_n^2 - 3 \\ 0 & 1 & 4x_{n+1} & 12x_{n+1}^2 - 3 \\ 0 & 1 & 4x_{n+2} & 12x_{n+2}^2 - 3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} y_{n+2} \\ f_n \\ f_{n+1} \\ f_{n+2} \end{pmatrix}$$

Setting $\{x_n = 0, x_{n+1} = x_n + h, x_{n+2} = 2h\}$ and simplifying the resulting expression by MATHEMATICA SOFTWARE, we obtain the parameters.

$$a_0 = \frac{9f_n + 8h^2f_n - 12f_{n+1} + 32h^2f_{n+1} + 3f_{n+2} + 8h^2f_{n+2} - 24hy_{n+2}}{24h}$$

$$a_1 = \frac{-f_n + 8h^2f_n - 2f_{n+1} - f_{n+2}}{8h^2}$$

$$a_2 = \frac{3f_n + 4f_{n+1} + f_{n+2}}{8h}$$

$$a_3 = \frac{f_n}{24h^2} - \frac{f_{n+1}}{12h^2} + \frac{f_{n+2}}{24h^2}$$

If substituted into equation (5.3), we have

$$\frac{9f_n + 8h^2f_n - 12f_{n+1} + 32h^2f_{n+1} + 3f_{n+2} + 8h^2f_{n+2} - 24hy_{n+2}}{24h}$$

$$+ \frac{[f_n - 8h^2f_n + 2f_{n+1} - f_{n+2}][x_{n+2}]}{8h^2} + \frac{[3f_n - 4f_{n+1} + f_{n+2}][2x_{n+2}^2 - 1]}{8h}$$

$$+\left[\frac{f_n}{24h^2}-\frac{f_{n+1}}{12h^2}+\frac{f_{n+2}}{24h^2}\right]\left[4x_{n+2}^3-3x_{n+2}\right] \quad (5.8)$$

When fully simplified by MATHEMATICA SOFTWARE, we have

$$y_{n+3}=y_{n+2}+\frac{1}{12}h\left[5f_n-16f_{n+1}+23f_{n+2}\right]$$

## 6. DERIVATION OF ADAMS- BASHFORTH METHOD WHEN K= 4

$$y_{n+k}=a_{n+k-1}y_{n+k-1}+h\sum_{j=0}^{k-1}b_jf_{n+j}$$

$$y_{n+4}=a_{n+3}y_{n+3}+h\sum_{j=0}^{3}b_jf_{n+j} \quad (6.1)$$

$$Note: y(x)=\sum_{j=0}^{4}a_jT_j(x) \quad (6.2)$$

$$\Rightarrow y(x)=a_0T_0(x)+a_1T_1(x)+a_2T_2(x)+a_3T_3(x)+a_4T_4(x) \quad (6.3)$$

$$y(x)=a_0+a_1x+a_2[2x^2-1]+a_3[4x^3-3x]+a_4[8x^4-8x^2+1]$$

when $x=x_{n+3}$, we have

$$y(x_{n+3})=a_0+a_1x_{n+3}+a_2[2x_{n+3}^2-1]+a_3[4x_{n+3}^3-3x_{n+3}]+a_4[8x_{n+3}^4-8x_{n+3}^2+1] \quad (6.4)$$

$$y'(x_n)=f_n=a_1(x)+a_2[4x]+a_3(12x^2-3)+a_4(32x^3-16x) \quad (6.5)$$

When $x=x_n$ in (6.5)

$$y'(x_n)=f_n=a_1(x)+a_2[4x_n]+a_3(12x_n^2-3)+a_4(32x_n^3-16x_n) \quad (6.6)$$

When $x=x_{n+1}$ in (6.5), we have

$$y'(x_{n+1})=f_{n+1}=a_1(x_{n+1})+a_2[4x_{n+1}]+a_3(12x_{n+1}^2-3)+a_4(32x_{n+1}^3-16x_{n+1}) \quad (6.7)$$

When $x=x_{n+2}$ in (6.5), we have

$$y'(x_{n+2})=f_{n+2}=a_1(x_{n+2})+a_2[4x_{n+2}]+a_3(12x_{n+2}^2-3)+a_4(32x_{n+2}^3-16x_{n+2}) \quad (6.8)$$

When $x=x_{n+3}$ in (6.5), we have

$$y'(x_{n+3})=f_{n+3}=a_1(x_{n+3})+a_2[4x_{n+3}]+a_3(12x_{n+3}^2-3)+a_4(32x_{n+3}^3-16x_{n+3}) \quad (6.9)$$

$$\begin{pmatrix} 1 & x_{n+3} & 4x_{n+3}^3-3x_{n+3} & 8x_{n+3}^4-8x_{n+3}^2+1 \\ 0 & 4x_n & 12x_n^2-3 & 32x_n^3-16x_n \\ 0 & 4x_{n+1} & 12x_{n+1}^2-3 & 32x_{n+1}^3-16x_{n+1} \\ 0 & 4x_{n+2} & 12x_{n+2}^2-3 & 32x_{n+2}^3-16x_{n+2} \\ 0 & 4x_{n+3} & 12x_{n+3}^2-3 & 32x_{n+3}^3-16x_{n+3} \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}=\begin{pmatrix} y_{n+3} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix}$$

Setting $x_{n+h}=h, x_{n+2}=x_n+2h, x_{n+3}=x_n+3h, x_n=0$ and simplifying the result expression by MATHEMATICA SOFTWARE, we obtain the parameters.

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$$\frac{3f_n+88h^2f_n-9f_{n+1}-144h^2f_{n+1}216h^4f_{n+1}+9f_{n+2}+72h^2f_{n+2}+216h^4f_{n+2}-3f_{n+3}-16h^2f_{n+3}+72h^4f_{n+3}}{192h^3}-$$

$$+\left[\frac{-2f_n-8h^2f_n+5f_{n+1}-4f_{n+2}+f_{n+3}}{8h^2}\right][x_{n+3}]$$

$$+\left[\frac{[f_n-22h^2f_n-3f_{n+1}-36h^2f_{n+2}+3f_{n+2}+18h^2f_{n+2}-f_{n+3}-4h^2f_{n+3}-4h^2f_{n+3}][2x_{n+3}^2-1]}{48h^3}\right]$$

$$a_3=\frac{f_n}{12h^2}-\frac{5f_{n+1}}{24h^2}+\frac{f_{n+2}}{6h^2}-\frac{f_{n+3}}{24h^2}$$

$$a_4=\frac{f_n-3f_{n+1}+3f_{n+2}-2f_{n+3}}{192h^3}$$

If substituted into equation (6.4), we have

$$\frac{3f_n+88h^2f_n-9f_{n+1}-144h^2f_{n+1}216h^4f_{n+1}+9f_{n+2}+72h^2f_{n+2}+216h^4f_{n+2}-3f_{n+3}-16h^2f_{n+3}+72h^4f_{n+3}-192h^3f_{n+3}}{192h^3}$$

$$+\left[\frac{-2f_n-8h^2f_n+5f_{n+1}-4f_{n+2}+f_{n+3}}{8h^2}\right][x_{n+3}]$$

$$+\left[\frac{[f_n-22h^2f_n-3f_{n+1}-36h^2f_{n+2}+3f_{n+2}+18h^2f_{n+2}-f_{n+3}-4h^2f_{n+3}-4h^2f_{n+3}][2x_{n+3}^2-1]}{48h^3}\right]$$

$$+\frac{f_n}{12h^2}-\frac{5f_{n+1}}{24h^2}+\frac{f_{n+2}}{6h^2}-\frac{f_{n+3}}{24h^2}\left[4x_{n+3}^3-3x_{n+3}\right]+\frac{f_n-3f_{n+1}+3f_{n+2}-f_{n+3}}{192h^3}\left[8x_{n+3}^4-8x_{n+3}^2+1\right]$$

when fully simplified by MATHEMATICA SOFTWARE, we have

$$y_{n+4}=y_{n+3}+\frac{h}{24}[-9f_n+37f_{n+1}-59f_{n+2}+55f_{n+3}]$$

## 7. DERIVATION OF ADAMS-BASHFORTH METHOD WHEN K=5

$$y(x)=\sum_{j=0}^{4}a_jT_j(x) \tag{7.2}$$

$$y(x)=a_0T_0(x)+a_1T_1(x)+a_2T_2(x)+a_3T_3(x)+a_4T_4(x)+a_5T_5(x)$$

$$y(x)=a_0+a_1x+a_2[2x^2-1]+a_3[4x^3-3x]+a_4[8x^4-8x^2+1]+a_5[16x^5-20x^3+5x] \tag{7.3}$$

When $x=x_{n+4}$ in (7.3), we have

$$y(x_{n+4})=a_0+a_1x_{n+4}+a_2[2x_{n+4}^2-1]+a_3[4x_{n+4}^3-3x_{n+4}]+a_4[8x_{n+4}^4-8x_{n+4}^2+1]$$
$$+a_5[16x_{n+4}^5-20x_{n+4}^3+5x_{n+4}] \tag{7.4}$$

$$y'(x)=f_n=a_1+a_2 4x+a_3(12x^2-3)$$
$$+a_4[32x^3-16]+a_5[80x^4-60x^2+5] \tag{7.5}$$

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when $x = x_n$ in (7.5), we have

$$y'(x_n) = f_n = a_1 + a_2 4x_n + a_3(12x^2 - 3) + a_4[32x^3 - 16] + a_5[80x^4 - 60x^2 + 5] \tag{7.6}$$

when $x = x_{n+1}$ in (7.5), we have

$$y(x_{n+1}) = f_{n+1} = a_1 + a_2 4x_{n+1} + a_3(12x_{n+1}^2 - 3)$$
$$+ a_4[32x_{n+1}^3 - 16_{n+1}] + a_5[80x_{n+1}^4 - 60x_{n+1}^2 + 5] \tag{7.7}$$

when $x = x_{n+2}$ in (7.5), we have

$$y'(x_{n+2}) = f_{n+2} = a_1 + a_2 4x_{n+2} + a_3(12x_{n+2}^2 - 3)$$
$$+ a_4[32x_{n+2}^3 - 16_{n+2}] + a_5[80x_{n+2}^4 - 60x_{n+2}^2 + 5] \tag{7.8}$$

when $x = x_{n+3}$ in (7.5), we have

$$y'(x_{n+3}) = f_{n+3} = a_1 + a_2 4x_{n+3} + a_3(12x_{n+3}^2 - 3) \tag{7.9}$$

When X= $X_{n+4}$ in (7.5), we have

$$y'(x_{n+4}) = f_{n+4} = a_1 + a_2 4x_{n+4} + a_3(12x_{n+4}^2 - 3)$$
$$+ a_4[32x_{n+4}^3 - 16_{n+2}] + a_5[80x_{n+4}^4 - 60x_{n+4}^2 + 5] \tag{7.10}$$

$$\begin{pmatrix} 1 & x_{n+4} & 2x_{n+4}^2 - 14x_{n+4}^3 - 3x_{n+4} & 8x_{n+4}^4 - 8x_{n+4}^2 + 116x_{n+4}^5 - 20x_{n+4}^3 + 5x_{n+4} & & \\ 0 & 1 & 4x_n & 12x_n^2 - 3 & 32x_n^3 - 16x_n & 80x_n^4 - 60x_n^2 + 5 \\ 0 & 1 & 4x_{n+1} & 12x_{n+1}^2 - 3 & 32x_{n+1}^3 - 16x_{n+1} & 80x_{n+1}^4 - 60x_{n+1}^2 + 5 \\ 0 & 1 & 4x_{n+2} & 12x_{n+2}^2 - 3 & 32x_{n+2}^3 - 16x_{n+2} & 80x_{n+2}^4 - 60x_{n+2}^2 + 5 \\ 0 & 1 & 4x_{n+3} & 12x_{n+3}^2 - 3 & 32x_{n+3}^3 - 16x_{n+3} & 80x_{n+3}^4 - 60x_{n+3}^2 + 5 \\ 0 & 1 & 4x_{n+4} & 12x_{n+4}^2 - 3 & 32x_{n+4}^3 - 16x_{n+4} & 80x_{n+4}^4 - 60x_{n+4}^2 + 5 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} y_{n+4} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix}$$

$$a_1 = \frac{f_n - 70h^2 f_n - 192h^2 f_n + 4f_{n+1} - 208h^2 f_{n+1} - 6f_{n+2} - 288h^2 f_{n+2} + 4f_{n+3} + 112h^2 f_{n+3} - f_{n+4} - 22h^2 f_{n+4}}{192h^4}$$

$$a_2 = \frac{5f_n + 50h^2 f_n - 18f_{n+1} - 96h^2 f_{n+1} + 24f_{n+2} + 72h^2 f_{n+2} - 14f_{n+3} - 32h^2 f_{n+3} + 3f_{n+4} + 6h^2 f_{n+4}}{96h^3}$$

$$a_3 = \frac{3f_n - 140h^2 f_n + 12f_{n+1} + 416h^2 f_{n+1} - 18f_{n-2} - 456h^2 f_{n+2} + 12f_{n+3} + 224h^2 f_{n-3} - 3f_{n-4} - 44h^2 f_{n+4}}{1152h^4}$$

$$a_4 = \frac{5f_n - 18f_{n+1} + 24f_{n+2} - 14f_{n+3} + 3f_{n+4}}{384h^3}$$

$$a_3 = \frac{f_n + 4f_{n+1} - 6f_{n+2} + 4f_{n+3} - f_{n+4}}{1920h^4}$$

If substituted into equation (7.3) and fully simplified by MATHEMATICA SOFTWARE, we have;

$$y_{n+5} = y_{n+4} + \frac{h}{720}[251f_n - 1274f_{n+1} + 2616f_{n+2} - 2774f_{n+3} + 1901f_{n+4}]$$

## 8. IMPLEMENTATION

Apply Adam Bashforth methods for $k = 1, 2, 3, 4 and 5$ to the following initial value problem, choosing $h = 0.2$ and computing $y_1, \ldots y_{10}$

$$y^1 = x + y, y(0) = 0$$

Solution : Euler's method (Adams- Bashforth for $k = 1$)

$$y_{n+1} = y_n + hf_n$$

Here $f(x, y) = x + y$

$$\Rightarrow y_{n-1} = y_n + 0.2(x_n + y_n)$$

Exact solution

$$y = e^x - x - 1$$

## 9. NUMERICAL SOLUTION

$$y_1 = y_0 + 0.2(x_0 + y_0) = 0 + 0.2\ (0{+}0) = 0.0000$$

$$y_2 = y_2 + 0.2(x_1 + y_1) = 0.0000 + 0.2\ (0.2{+}0.0000) = 0.0400$$
$$y_3 = y_3 + 0.2(x_2 + y_2) = 0.0400{+}0.2\ (0.4{+}0.0400){=}\ 0.1280$$
$$y_4 = y_4 + 0.2(x_3 + y_3) = 0.1280{+}0.2\ (0.6{+}0.1280) = 0.2736$$
$$y_5 = y_5 + 0.2(x_4 + y_4) = 0.2736{+}0.2\ (0.8{+}0.2736) = 0.4883$$
$$y_6 = y_6 + 0.2(x_5 + y_5) = 0.4883 + 0.2(1.0{+}0.4883) = 0.7860$$
$$y_7 = y_7 + 0.2(x_6 + y_6) = 0.7860{+}0.2\ (1.2{+}0.7860){=}\ 1.1832$$
$$y_8 = y_8 + 0.2(x_7 + y_7) = 1.1832{+}0.2(1.4 + 1.1832) = 1.7000$$
$$y_9 = y_9 + 0.2(x_8 + y_8) = 1.7000{+}0.2(1.6{+}1.7000){=}2.3600$$
$$y_{10} = y_{10} + 0.2(x_9 + y_9) = 2.3600{+}0.2\ (1.8{+}2.3600){=}3.1920$$

## 10. EXACT SOLUTION

$$y(x) = e^x - x - 1$$

$$y_0 = e^0 - 0 - 1 = 0.0000$$
$$y_1 = e^1 - 1 - 1 = 0.0214$$
$$y_2 = e^2 - 2 - 1 = 0.0918$$
$$y_3 = e^3 - 3 - 1 = 0.2221$$
$$y_4 = e^4 - 4 - 1 = 0.4255$$
$$y_5 = e^5 - 4 - 1 = 0.7183$$
$$y_6 = e^6 - 6 - 1 = 1.1201$$
$$y_7 = e^7 - 7 - 1 = 1.6552$$
$$y_8 = e^8 - 8 - 1 = 2.3530$$
$$y_9 = e^9 - 9 - 1 = 3.2496$$
$$y_{10} = e^{10} - 10 - 1 = 4.3891$$

**Table 1: Adams Bashforth Method for k=1** $[y_{n+1} = y_n + hf_n]$

| N | $x_n$ | $y_n$ | Exact values | Error |
|---|---|---|---|---|
| 0 | 0.0 | 0.0000 | 0.0000 | 0.0000 |
| 1 | 0.2 | 0.0000 | 0.0214 | 0.0214 |
| 2 | 0.4 | 0.0400 | 0.0918 | 0.0518 |
| 3 | 0.6 | 0.1280 | 0.2221 | 0.0941 |
| 4 | 0.8 | 0.2736 | 0.4255 | 0.1519 |
| 5 | 1.0 | 0.4883 | 0.7183 | 0.2300 |
| 6 | 1.2 | 0.7860 | 1.1201 | 0.3341 |
| 7 | 1.4 | 1.1832 | 1.6552 | 0.4720 |
| 8 | 1.6 | 1.7000 | 2.3530 | 0.6530 |
| 9 | 1.8 | 2.3600 | 3.2496 | 0.8896 |
| 10 | 2.0 | 3.1920 | 4.3891 | 1.1971 |

**Table 2: Adams Bashforth method for K =2**

$$[y_{n+2} = y_{n+1} + \frac{h}{2}[3f_{n+1} - f_n]$$

| N | $x_n$ | $y_n$ | Exact values | Error |
|---|---|---|---|---|
| 0 | 0.0 | 0.0000 | 0.0000 | 0.0000 |
| 1 | 0.2 | 0.0214 | 0.214 | 0.0000 |
| 2 | 0.4 | 0.0878 | 0.0918 | 0.0040 |
| 3 | 0.6 | 0.2158 | 0.2221 | 0.0063 |
| 4 | 0.8 | 0.4169 | 0.4255 | 0.0086 |
| 5 | 1.0 | 0.7065 | 0.7183 | 0.0118 |
| 6 | 1.2 | 1.1041 | 1.1201 | 0.0160 |
| 7 | 1.4 | 1.6337 | 1.6552 | 0.0215 |
| 8 | 1.6 | 2.3244 | 2.3530 | 0.0286 |
| 9 | 1.8 | 3.2118 | 3.2496 | 0.0378 |
| 10 | 2.0 | 4.3393 | 4.3891 | 0.1878 |

**Table 3: Adams Bashforth method for K =3** $\left[ y_{n+3} = y_{n+2} + \frac{h}{2}[5f_n - 16f_{n+1} + 23f_{n+2}] \right]$

| N | $x_n$ | $y_n$ | Exact values | Error |
|---|---|---|---|---|
| 0 | 0.0 | 0.0000 | 0.0000 | 0.0000 |
| 1 | 0.2 | 0.0214 | 0.214 | 0.0000 |
| 2 | 0.4 | 0.0878 | 0.0918 | 0.0040 |
| 3 | 0.6 | 0.2158 | 0.2221 | 0.0063 |
| 4 | 0.8 | 0.4169 | 0.4255 | 0.0086 |
| 5 | 1.0 | 0.7065 | 0.7183 | 0.0118 |
| 6 | 1.2 | 1.1041 | 1.1201 | 0.0160 |
| 7 | 1.4 | 1.6337 | 1.6552 | 0.0215 |
| 8 | 1.6 | 2.3244 | 2.3530 | 0.0286 |
| 9 | 1.8 | 3.2118 | 3.2496 | 0.0378 |
| 10 | 2.0 | 4.3393 | 4.3891 | 0.0498 |

**Table 4: Adams Bashforth method K= 4** $\left[ y_{n+4} = y_{n+3} + \frac{h}{24}[9f_n - 37f_{n+1} + 59f_{n+2} + 55f_{n+3}] \right]$

| N | $x_n$ | $y_n$ | Exact values | Error |
|---|---|---|---|---|
| 0 | 0.0 | 0.0000 | 0.0000 | 0.0000 |
| 1 | 0.2 | 0.0214 | 0.214 | 0.0000 |
| 2 | 0.4 | 0.0878 | 0.0918 | 0.0040 |
| 3 | 0.6 | 0.2158 | 0.2221 | 0.0063 |
| 4 | 0.8 | 0.4181 | 0.4255 | 0.0074 |
| 5 | 1.0 | 0.7091 | 0.7183 | 0.0092 |
| 6 | 1.2 | 1.1085 | 1.1201 | 0.0162 |
| 7 | 1.4 | 1.6406 | 1.6552 | 0.0146 |
| 8 | 1.6 | 2.3348 | 2.3530 | 0.0182 |
| 9 | 1.8 | 3.2269 | 3.2496 | 0.0227 |
| 10 | 2.0 | 4.3607 | 4.3891 | 0.0248 |

**Table 5: Adams Bashforh method for K =5** $\left[ y_{n+5} = y_{n+4} + \frac{h}{720}[251f_n - 1274f_{n+1} + 2616f_{n+2} + 2774f_{n+3} + 1901f_{n+4}] \right]$

| N | $x_n$ | $y_n$ | Exact values | Error |
|---|---|---|---|---|
| 0 | 0.0 | 0.0000 | 0.0000 | 0.0000 |
| 1 | 0.2 | 0.0214 | 0.0214 | 0.0000 |
| 2 | 0.4 | 0.0878 | 0.918 | 0.0040 |
| 3 | 0.6 | 0.2158 | 0.2221 | 0.0063 |
| 4 | 0.8 | 0.4181 | 0.4255 | 0.0074 |
| 5 | 1.0 | 0.7088 | 0.7183 | 0.0095 |
| 6 | 1.2 | 1.1082 | 1.1201 | 0.0119 |
| 7 | 1.4 | 1.6408 | 1.6552 | 0.0144 |
| 8 | 1.6 | 2.3355 | 2.3530 | 0.0175 |
| 9 | 1.8 | 3.2281 | 3.2496 | 0.0215 |
| 10 | 2.0 | 4.3626 | 4.3891 | 0.0265 |

**Table 6: Adam Bashforth Methods for k=3, 4, and 5**

| N | Exact Solution | K=3 | K=4 | K=5 |
|---|---|---|---|---|
| 0.0 | 2.000000000 | 2.000000000 | 2.000000000 | 2.000000000 |
| 0.1 | 1.732778818 | 1.732778818 | 1.732778818 | 1.732778818 |
| 0.2 | 1.523960081 | 1.523960081 | 1.523960081 | 1.523960081 |
| 0.3 | 1.360420363 | 1.359628069 | 1.359628069 | 1.359628069 |
| 0.4 | 1.236235687 | 1.23518843 | 1.236029608 | 1.236029608 |
| 0.5 | 1.143789022 | 1.142345344 | 1.4334805 | 1.143016003 |
| 0.6 | 1.077089871 | 1.075527538 | 1.077038936 | 1.077053987 |
| 0.7 | 1.031544687 | 1.029935038 | 1.03140519 | 1.030851595 |
| 0.8 | 1.003318906 | 1.001738505 | 1.003391861 | 1.003463842 |
| 0.9 | 0.989273054 | 0.987760793 | 0.989275575 | 0.98857031 |
| 1.0 | 0.986836745 | 0.985421503 | 0.986953254 | 0.987153234 |
| 1.1 | 0.993905527 | 0.992601262 | 0.993969735 | 0.993158432 |

**Table 7: Error from Adams Bashforth methods for k=3,4 and 5**

| N | K=3 | K=4 | K=5 |
|---|---|---|---|
| **0.0** | 0.000000000 | 0.000000000 | 0.000000000 |
| **0.1** | 0.000000000 | 0.000000000 | 0.000000000 |
| **0.2** | 0.000000000 | 0.000000000 | 0.000000000 |
| **0.3** | $7.92294 \times 10^{-4}$ | $7.92294 \times 10^{-4}$ | $7.92294 \times 10^{-4}$ |
| **0.4** | $1.37257 \times 10^{-3}$ | $2.96079 \times 10^{-4}$ | $2.96079 \times 10^{-4}$ |
| **0.5** | $1.443678 \times 10^{-3}$ | $4.40972 \times 10^{-4}$ | $7.73019 \times 10^{-4}$ |
| **0.6** | $1.562333 \times 10^{-3}$ | $5.0935 \times 10^{-5}$ | $3.5884 \times 10^{-5}$ |
| **0.7** | $1.609649 \times 10^{-3}$ | $1.39497 \times 10^{-4}$ | $6.93092 \times 10^{-4}$ |
| **0.8** | $1.580401 \times 10^{-3}$ | $-7.2955 \times 10^{-5}$ | $-1.44936 \times 10^{-4}$ |
| **0.9** | $1.512261 \times 10^{-3}$ | $-2.521 \times 10^{-6}$ | $7.02744 \times 10^{-4}$ |
| **1.0** | $1.415242 \times 10^{-3}$ | $-1.16509 \times 10^{-4}$ | $-3.16489 \times 10^{-4}$ |
| 1.1 | $1.304265 \times 10^{-3}$ | $-6.4208 \times 10^{-5}$ | $7.47095 \times 10^{-4}$ |

## CONCLUSION

The Adams Bashforth methods of k=4 and k=5 were shown to have lower errors than that of k=3 and k=2.

This paper demonstrated clearly that Adams Bashforth methods can be derived through Chebyshev polynomials which can be used to solve general ordinary differential equations.

The results of the numerical solutions for different values of x agree closely with the analytic solutions . Moreover, all the methods show small errors and the solution values agree closely with the exact solution (see Tables 1-5) .

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