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ANALYTIC SOLUTION TO A SIMPLY SUPPORTED ELASTIC RECTANGULAR PLATE WITH ADDITIONAL INTERNAL SUPPORT

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ABSTRACT

The mechanics of beams, shells and plates are of utmost importance in structural engineering, hence the need to study problems on plates and its solutions as a case study. The problem of a laterally loaded rectangular plate, simply supported along all its edges with an additional support along a line segment in the internal portion of the plate is considered. The problem is further extended by the introduction of an additional support along part of its center line,the transverse load distribution is varied giving various results according to the loading conditions. This give rise to an additional constraint and the resulting solution led to the determination of the deflection in the form of superposition integrals in terms of hyperbolic and sinusoidal functions.

Keywords: Plate, deflection, elasticity, series, load.

1.0 INTRODUCTION

The development of the mathematical theory of elasticity was of great deal to such nineteenth century scientists like Cauchy, Navier and Green (Timoshenko and Goodier, 1951). Plates are widely used in structured element in most engineering works. The practical project, the intermediate supports, such as elastic foundation etc. in one or two direction maybe used to improve the load bearing capacity of the plates (Momoh,1997). The bending properties of plates depend greatly on its thickness as compared with its other dimensions. You could have thin plates with small deflections (Abubakar and Hussein, 2017), thin plates with large deflections and thick plates. Here we would be concerned with problems involving thin plates with small deflections.
The first solution of the problem of bending of supported rectangular plates and the use of double trigonometric series was due to Navier (Timoshenko and Goodier, 1951), who presented a paper on this subject to the French Academy in the early nineteenth century.
The series-based solution is one of the most common in early established methods for the analysis of plates, but less effort has been given to the analysis of internally supported plates. (Momoh, 1997), these series based solution have been well summarized in (Timoshenko and Goodier,1951),(Abubakar and Hussain,2017) based on Navier and Levy methods. The usual problem involving a thin elastic plate with flexural rigidity D carrying a transverse load q(x, y) leads to the determination of the solution of the differential equation

$$\frac{d^4w}{dx^4} + 2\frac{d^4w}{dx^2dy^2} + \frac{d^4w}{dy^4} = \frac{q(x,y)}{D}$$

for the deflection w(x, y). The boundary conditionsare generally prescribed according to the type of support along the edges of the plate. Analysis of internally supported plates has been covered in (Timoshenko and Woinesky-kreger, 1959) for the assumptions that the layout is regular and columns are to be treated as point reactions. This later assumptions gives an infinite bending movement at the point of the support leading to approximation procedure to treat singularity of the bending moment at the center of the support as given in (Timoshenko and Woinesky-kreger, 1959) . This

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approximation, however, leads to maximum bending moment at the center of the support instead of the face of the support (Momoh, 1997),(Ephriam and Orumu,2014).
However, the case being considered has an additional support along part of its center line. With the internal support, the deflection is constrained along a line segment in the internal support intersects one edge at the boundary or situated in the middle of the plate, only half or a quarter domain needs to be considered under symmetrical loading conditions of equation (1) with mixed boundary conditions. Since the boundary containing the line of internal support is made of two parts (the line of internal support and the line of symmetry), the mixed boundary conditionsuch a boundary are in pairs. In this paper we have chosen the internal support that it does not intersect one edge and is not necessarily centered (Abubakar and Hussain,2017). The reaction on the plates from internal support is considered as unknown line load is formulated by superposing the solution corresponding to a point load on the plate.
The analytical solution to the plate problem is obtain from the formulated governing equation in relation to varying applied loads and additional internal support. Numerical method on the other hand, gives more flexibility for the layout of the internal support and supporting conditions of the plate(Momoh, 1997),(Slizard 2004). This give more reason why most structural engineers used finite element method of solutions instead ofthe analytical solution which is more accurate.However, in this paper we will focus our attention on the problems concerned with the bending of rectangular plates with an additional internal support along parts of its center line.

2.0 METHODOLOGY

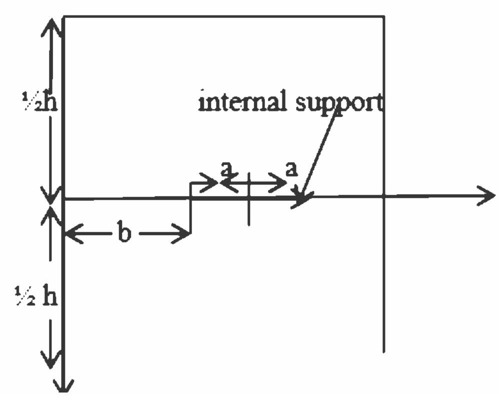

Figure 1:Showing simply supported rectangular plate with additional internal support.

The rectangular plate shown above has dimension h by l and h is laterally loaded by a general loading q(x, y),The edges are simply supported. The internal support is located along the x-axis from b-a to b+a, having a length of 2a. The internal support is at the same level as the edges and is assumed to permit no deflection under loading.

2.1 For the plate problems

We consider a simply supported rectangular plate subjected to a distributed load q(x, y) as shown in figure(1) above, uniform and concentrated load would be considered for a rectangular plate. The general governing equation is given by

$$\nabla^4 w = \frac{q(x,y)}{D} \qquad (2)$$

Where W is the deflection of the plate, q is the applied load and D is the flexural rigidity of the plate. See (Timoshenko,1956) for the derivation of equation (2).

The problem described above requires the solution of the differential equation.

$$\nabla^4 w = \frac{q}{D} \qquad (3)$$

w h the b c.

$$W=0 \; f \quad x = 0, x = l, y = \pm\frac{1}{2}h$$

$$\frac{d^2 w}{dx^2} = 0 \; f \quad x = 0, x = l \qquad (4)$$

$$\frac{d^2 w}{dy^2} = 0 \; f \quad y = \pm\frac{1}{2}h$$

and the associated internal constraint condition.

3.0 RESULT OF ANALYTICAL SOLUTION FOR THE PROBLEMS

3.1 Uniformly Loaded Rectangular Plate

We first considered the solution of the problem of equation (3) subject to equation (4). It will be assumed that the plate (l, h) is simply supported along all its edges and under the action of the distributed loads.

i. $Q= q_0 \sin\frac{\pi x}{l} \sin\frac{\pi}{h}$ (5)

ii. $Q=q_0 \sin\frac{\pi}{l} \cos\frac{\pi}{h}$ (6)

For both cases , we try the assumed solution, for case (I), we have

$$W=C\sin\frac{\pi}{l} \sin\frac{\pi}{h} \quad (7)$$

where C is a constant

Thus $$\frac{d^4w}{dx^4} = C(\pi_l)^4 \sin\frac{\pi}{l} \sin\frac{\pi}{h} \quad (8)$$

$$\frac{d^4w}{dy^4} = C\left(\frac{\pi}{l}\right)^2\left(\frac{\pi}{h}\right)^2 \sin\frac{\pi}{l} \sin\frac{\pi}{h} \quad (9)$$

Which when substituted into equation (3) we obtain

$$C[\left(\frac{\pi}{l}\right)^4 + 2\left(\frac{\pi}{l}\right)^2\left(\frac{\pi}{l}\right)^2 + \left(\frac{\pi}{h}\right)^4] = \frac{q_c}{L} \quad (10)$$

$$C=\frac{q_c}{L[\left(\frac{\pi}{l}\right)^2+\left(\frac{\pi}{h}\right)^2]^2} \quad (11)$$

Hence the solution is given as

$$W=\frac{q_c}{L[\left(\frac{\pi}{l}\right)^2+\left(\frac{\pi}{h}\right)^2]^2} \sin\frac{\pi}{l} \sin\frac{\pi}{h} \quad (12)$$

For case (II), we observed that the constant C assumes the same value as in (I)

However in the case where the supports are on two sides only (x=0)and x=l), the solution takes the form of series representation as

$$W= \sum_{m=1}^{\infty} y_m \sin\frac{m}{l} \quad (13)$$

Where y_m is a function of y only. Thus , is this case each term of equation (13), satisfies the boundary conditions W= 0 and d^2w/dx^2=0 on these two sides.

The aim is to determine y_m in such a form as to satisfy the boundary conditions on the sides $y=\pm h/2$

And is also the equation of deflection surface (3). In applying this method to uniformly loaded and simply supported rectangular plates, a further simplification can be made by taking the solution of equation (3) in the form

$W=W_1+ W_2$

$$W_1 = \frac{q}{2\,L}(x^4 - 2lx^3 + l^3x) \quad (15)$$

Where W_1 is the deflection of a uniformly loaded strip parallel to the x-axis. It satisfies equation (3) and also the boundary condition at the edges x=0 and x=l. The expression W_2 in the form of the equation (14) and substituting it into equation (1) we obtain

$$\sum_{m=1}^{\infty}(y_m^{ll} - 2\frac{m^2\pi^2}{l^2} y_m^1 + \frac{m^4\pi^4}{l}) = 0 \quad (16)$$

This equation can be satisfied for all values of x only if the function y_m satisfies the equation

$$y_m^{ll} - 2\frac{m^2\pi^2}{l^2} y_m^1 + \frac{m^4\pi^4}{l} y_m = 0 \quad (17)$$

The general integral of this equation takes the form (see Abubakar and Hussain,2017).

$$Y_m = \frac{ql^4}{L}(A_m \cosh\frac{m}{l} + B_m\frac{m}{l}\sinh\frac{m}{l} + C_m \sinh\frac{m}{l} + B_m\frac{m}{l}\sinh\frac{m}{l}) \sin\frac{m}{l} \quad (18)$$

In view of the fact that the deflection surface of the plate is symmetrical with respect to the x-axis, we can keep expression (1) only for even function of y and the integration constant $C_m= D_m$

Thus, the deflection W takes the form

$$W=\frac{q}{2\,L}(x^4 - 2lx^3 + l^3x) + \frac{ql^4}{L}\sum_{m-1}^{\infty}(A_m \cosh\frac{m}{l} + B_m\frac{m}{l}\sinh\frac{m}{l}) \sin\frac{m}{l} \quad (19)$$

Which satisfies equation (3) and also the boundary condition on the sides x= 0 and x = l. it is now necessary to determine the constants of integration A_m and B_m in such a manner as to satisfy the boundary condition, W= 0, $\frac{d^4w}{dy^4}$= 0 on sides y=$\pm h/2$ (20)

Equation (15) can be put in the form of a trigonometric series, as

$$\frac{q}{2L}(x^4 - 2lx^3l^3x) = \frac{4ql^4}{\pi^5 L}\sum_{m=1}^{\infty}\frac{1}{m^4}\sin\frac{m}{l} \quad (21)$$

Where m = 1,3,5

Thus equation (19) can now be represented in the form

$$W=\frac{ql^4}{D}\sum_{m=1}^{\infty}\left(\frac{4}{\pi^5 m^5}+A_m\cosh\frac{m}{l}+B_m\frac{m}{l}\sinh\frac{m}{l}\right)\sin\frac{m}{l} \quad (22)$$

By substituting this expression in equation (20) we obtain the following equation for determining the constants A_m and B_m

$$\frac{4}{\pi^5 m^5}+A_m\cosh a \quad + u_m B_m \sinh u_m = 0 \quad (23)$$

Where $a_m = \frac{m}{2l}$

Hence

$$A_m=-2\frac{(u_m t_i \quad u_m+2)}{\pi^5 m^5 c \quad u_m}, B_m = \frac{2}{\pi^5 m^5 c \quad u_m} \quad (24)$$

Substituting these values of the constants in equation for deflection, we obtain the deflection W of the plate as

$$W=\frac{4ql^4}{\pi^5 L}\sum_{m=1}^{\infty}\frac{1}{m^5}\left[1-\frac{u_m t \quad hu_m+2}{2c \quad hu_m}\cosh\left(\frac{2u_m y}{h}\right)+\frac{u_m}{2c \quad u_m}\frac{2y}{h}\sinh\left(\frac{2u_m y}{h}\right)\right]\sin\frac{m}{l} \quad (25)$$

From which the deflection at any point can be calculated by using the tables of hyperbolic function .

The maximum deflection is obtained at the middle of the plate $(\frac{l}{2}, y=0)$ as

$$W_{max}=\frac{4ql^4}{\pi^5 L}\sum_{m=1,3,5}^{\infty}\frac{(-1)^{\frac{m-1}{2}}}{m^5}\left(1-\frac{u_m t \quad hu_m+2}{2c \quad hu_m}\right) \quad (26)$$

3.2 Concentrated Load On A Simply Supported Rectangular Plate

In the case of a plate carrying a single load p at a given point (ʒ,η), we can obtain the solution of the deflection in the form of a double trigonometric series representation, (Szilard 2004) , thus

$$W==\frac{4ph^2}{\pi^4 l}\sum_{m=1}^{\infty}T_m\sin\frac{m}{l}\sin\frac{m}{l} \quad (27)$$

Where $T_m = \ = \sum_{m=1}^{\infty} \frac{s.\ \frac{m}{n}s.\ \frac{m}{n}}{\left(\frac{m^2 n^2}{h}+n^2\right)^2}$ (28)

Equation (28) can be written as

$$\left.\begin{aligned}T_{m_1}&=\sum_{n=1}^{\infty}\frac{c\left(\frac{n(y-\eta)}{h}\right)}{\left(\frac{m^4h^4}{h}+n^4\right)^4},\\ T_{m_2}&=\sum_{n=1}^{\infty}\frac{c\left(\frac{n(y+\eta)}{h}\right)}{\left(\frac{m^4h^4}{l^4}+n^4\right)^4},\\ T_m&=\sum_{n=1}^{\infty}\frac{c\left(\frac{n(y+\eta)}{h}\right)}{\left(\frac{m^4h^4}{l^4}+n^4\right)^4}\end{aligned}\right\} \quad (29)$$

Following the same evaluation method as in simply supported we obtain for concentrated load for the deflection of the plate as Szilard (2004).

$$W=\frac{p}{4 \quad L}\sum_{m=1}^{\infty}\left[C_1+u_m t_i \quad hu_m\right)\sinh\left(\frac{u_m}{n}\right)(h-2y)-\left(\frac{u_m}{n}\right)(h-2y)\cosh\left(\frac{u_m}{h}\right)(h2y)]\ \sin\frac{\frac{m}{l}3_s\ \frac{m}{l}}{m^3 c \quad u_m}$$

(30)

Which is valid for y ≥ 0

Putting y=0,we obtain the deflection of the plate along the x-axis in the form

$$(W)_y=0=\frac{pl^2}{4\pi^3 L}\sum_{m=1}^{\infty}[t_i \quad ha_m - a_m s_i \quad h^4 a_m)(\sin\frac{\frac{m}{l}3_s\ \frac{m}{l}}{m^3}) \quad (31)$$

It is believed that series converges since the load is at the center of the plate.

3.3 Rectangular Plate With Internal Support In Terms Of Deflection

For our problem the internal support is located along x-axis from b-a to b+a, having a length of 2a. the aim is to solve equation (3) subject to the boundary condition 4 and the internal constraint condition. W= 0 for y =0, b – a ≤ x ≤ b + a consider a line load p(x) with known distribution applied along the line of internal support. If an infinitesimal element of the line load p(ʒ), y(ʒ), produces a deflection dw, (x, y) for the same plate such solution will be obtained from the solution of a simply supported rectangular plate under concentrated load. In this case along x-axis , dw(x,0) takes the form

$$Dw_1 \quad (\bar{x},\ 0) = \frac{l^4 p(\mathfrak{z})}{2\pi^4 L} \sum_{m=1}^{\infty} [t \quad h\alpha_m - \alpha_m s \quad h^2 \alpha_m) \frac{s \quad m \quad \mathfrak{z}}{l} \sin\frac{m}{l} \qquad (32)$$

By translating the origin to (b, 0) and normalizing the coordinates with respect to a, we let $\bar{x} = \frac{x-b}{a}, y = \frac{y}{a}\ \mathfrak{z} = \frac{\mathfrak{z}-b}{a}$ and so the internal support takes the position $\bar{y} = 0\,a \quad -1 \leq x \leq 1$ with the line p(x) distributed along the internal support, a deflection $W_2(x, y)$ will be produced and this is obtained by superposition integral of the form (Momoh, 1997)

$$W_1(\bar{x},0) = \frac{l^4 a}{2\pi^4 L} \int_{-1}^{+1} \sum_{M}^{\infty} \frac{(t \quad h\alpha_m \quad \alpha_m s \quad h^2 \alpha_m) s \quad [\beta_m(\bar{x}+\frac{b}{a})]}{m^4} \sin \left(\frac{b}{a}\right) p(\mathfrak{z}), d(\mathfrak{z}) \qquad (33)$$

$p(\mathfrak{z}) = e^{-c\mathfrak{z}}$, c > 0 and putting $\frac{l^4 a}{2\pi^4 L} = k$, we have

$$W_1(\bar{x},0) = K\int_{-1}^{+1} \sum_{m=1}^{\infty} \frac{(t \quad h\alpha_m - \alpha_m s \quad h^2 \alpha_m)]}{m^4} \sin\beta_m(\bar{x} + \frac{b}{a}) \sin\beta_m(\mathfrak{z} - \frac{b}{a})\, e^{-c\mathfrak{z}} d\mathfrak{z} \qquad (34)$$

$\mathfrak{z}$ + b/a =4 and $\tau = \bar{x} + b/a$

$$W_1(\bar{x},0) = K\int_{1+\frac{b}{a}}^{+1+\frac{b}{a}} \sum_{m=1}^{\infty} \frac{(t \quad h\alpha_m \quad \alpha_m s \quad h^4 \alpha_m)]}{m^4} \sin\beta_m \mathfrak{z} \sin\beta_m 4\; e^{-c(4-b/a)} \qquad (35)$$

$$= Ke^{cb/a} \sum_{m=1}^{\infty} \frac{(t \quad h\alpha_m \quad \alpha_m s \quad h^4 \alpha_m)]}{m^4} \sin\beta_m \mathfrak{z} \int_{1+\frac{b}{a}}^{+1+\frac{b}{a}} s \quad \beta_m \varphi e^{-c4} d \qquad (36)$$

By using integral calculus, we see after simplification that

$$\int \sin n \quad e^{-c} \quad d \quad = \frac{-e^{-c}}{1+\frac{n^4}{c^4}} \left[\frac{1}{c}\sin n \quad + \frac{n}{c^4}\cos n \quad\right]' \qquad (37)$$

For the studied problem, put x = 4 and n =β_m

Thus ,

$$\int_{-1+\frac{b}{a}}^{+1+\frac{b}{a}} s \quad \beta_m \varphi e^{-c4} d$$

$$= \frac{-e^{-c4}}{c^4\left(1+\frac{\beta^4 m}{c^4}\right)} C \sin\beta_m\varphi \quad \int_{1+\frac{b}{a}}^{+1+\frac{b}{a}}$$

$$= \frac{e^{-c4} C s \quad \beta_m\varphi \ c \quad \beta_m\varphi}{c^4+\beta_m^4} \quad \int_{-1+\frac{b}{a}}^{+1+\frac{b}{a}} \qquad (38)$$

Which after simplification and substitution to obtain the deflection, the result is

$$W_1(\bar{x},0) = K\sum_{m=1}^{\infty} \frac{(t \quad h\alpha_m \quad \alpha_m s \quad h^4 \alpha_m)]}{m^4(c^4+\beta_m^4)} \sin\beta_m \tau [e^{c}(C \quad \beta_m\left(\frac{b}{a} - 1\right) + \beta_m c \quad \beta_m\left(\frac{b}{a}-1\right)] - e^{-c}[C \quad \beta_m\left(1+\frac{b}{a}\right)]. \beta_m\left(\mathfrak{z} + \right. \qquad (39)$$

For the internal support illustration of special shapes and loading points can be considered as follow

$$\beta_m = \frac{m}{l}, \alpha_m = \frac{m \quad h}{2l}, \bar{x} = \frac{x-h}{a}, \tau = \bar{x} + \frac{b}{a} \qquad (40)$$

b = 3a, h =12a l= 8a, x =6a for values of C going from -100. The values of deflection (w) will be obtained in a numerical value to show its convergence.

4. DISCUSSION OF RESULTS

The paper deals with the investigation of the resultant effect on the deflection due to the loading condition on a rectangular plate. The rectangular plate, simply supported along its edges had line load p(x, y) in the form sinusoidal function and the problem of the resulting deflection was determined. By introducing the line load p(x) as an exponential function, the final deflection was determined in terms of geometry of the plate and the region of internal support. A Maximum deflection which is at its center was obtained when the plate had a concentrated load at the middle of the plate . The introduction of an additional material support led to the determination of the deflection was determined in terms of geometry of the plate and the region of internal support. A Maximum deflection which is at its center was obtained when the plate had a concentrated load at the middle of the plate . The introduction of an additional material support led to the determination of the deflection when the line load is of exponential form .The result obtained were in agreement with the result obtained by Momoh (1997), simply supported rectangular plate, Abubakar & Hussain(2017), series based solution for thin plates with rigid support. Zhou (1996),obtained analytical solution to vibration of rectangular plates simply supported at two opposite edge with arbitrary number of elastic line support and elastic foundation, but the result we obtain with internal support in the middle using analytical solution shows clear agreement with existing result. Further work will be done on shell and other material in terms of elastic behaviour.

5.0 CONCLUSION

The paper show clearly different analytical solutions for uniformly supported,concentrated load and internal support for a rectangular plate with its series solutions determined at different loading points.The series solutions are obtained in trigonometric form in terms of sinusoidal and hyperbolic functions.The deflections of loading are properly accounted for and numerical values assigned for further investigation. Further work is suggested on moving beams and shells.

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