COMPARISON OF EULER MARUYAMA AND MILSTEIN METHOD FOR THE NUMERICAL SOLUTION OF STOCHASTIC DIFFERENTIAL EQUATIONS.

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ABSTRACT
This paper presents an introduction to the concepts of numerical experiments involving stochastic differential equations (SDE) and the convergence of Euler-Maruyama method and Milstein method for the solution of stochastic differential equations (SDE). The Euler-Maruyama method is constructed within the ito integral framework. Milstein Method is derived from a truncation of the stochastic Taylor expansion of solution. The convergence of Euler-Maruyama and Milstein methods were compared.

Keywords: Stochastic differential equations, Euler-Maruyama method, Milstein method, Ito lemma

INTRODUCTION
Differential equations are used to describe the evolution of a system. Stochastic Differential Equations (SDEs) arise when a random noise is introduced into ordinary differential equations (ODEs).

We introduce Ito stochastic differential equations and explain what a solution is. It will turn out that stochastic differential equations are actually stochastic integral equations which involve ordinary and Ito stochastic integrals. We give a simple method for solving Ito stochastic differential equations. It is based on the Ito lemma. Stochastic differential equations which admit an explicit solution are the exception from the rule. Therefore numerical techniques for the approximation of the solution to a stochastic differential equation are called for. In what follows, such an approximation is called a numerical solution. Numerical solutions allow us to simulate as many sample paths as we want. A lot of authors has treated this topic among them are [9], [13], [14], [18] - [20].

The Ito lemma is the most important tool in Ito calculus. A first version of this fundamental result was proved by Ito (1951). Various versions of the Ito lemma and their proofs can be found in textbooks on stochastic calculus, for example, in [5], [10], [15] and [17].

We now have the necessary machinery in place to allow us to solve a stochastic differential equation numerically. A stochastic differential equation is a differential equation in which at least one term is a stochastic process. A general form for an SDE is:

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$$dX(t) = f(t,X(t)dt + g(t,X(t))dW(t) \quad (1)$$

$$X(0) = X_0$$

Which can be interpreted as an integral equation:

$$X(t) = X_0 + \int_0^t f(s,X(s))ds + \int_0^t g(s,X(s))dW(s) \quad (2)$$

However, in this report, we will consider autonomous SDEs, that is, SDEs that do not have any explicit dependence upon it:

$$X(0) = X_0$$

$$dX(t) = f(t,X(t))dt + g(t,X(t))dW(t)$$

$$X(t) = X_0 + \int_0^t f(s,X(s))ds + \int_0^t g(s,X(s))dW(s)$$

The first integral at the right hand side of equation (2) is called Riemann integral while the second integral is called Ito or stochastic integral. Many researchers have also worked on stochastic differential equations (SDEs) of the form (1), among these researchers are [1]-[4], [6], [7], [18], [21].

Derivation of Euler Maruyama Method

The Euler - Maruyama method is constructed within the ito integral framework. We will attempt to construct a numerical method from the integral form (2). We begin by setting $\delta t = \frac{T}{L}$ for some integer, L and let$\tau_i = i\delta t$ for $i = 0, \cdots, L$. We also define the numerical approximation to X_{τ_i} as X_i. Setting successively $t = \tau_i$ and $t = \tau_{i-1}$ in (2), we obtain:

$$X(\tau_i) = X_0 + \int_0^{\tau_i} f(X(s))ds + \int_0^{\tau_i} g(X(s))dW(s)$$

$$X(\tau_{i-1}) = X_0 + \int_0^{\tau_{i-1}} f(X(s))ds + \int_0^{\tau_{i-1}} g(X(s))dW(s)$$

If we subtract these equations, then we obtain:

$$X(\tau_i) = X_{\tau_{i-1}} + \int_{\tau_{i-1}}^{\tau_i} f(X(s))ds + \int_{\tau_{i-1}}^{\tau_i} g(X(s))dW(s) \quad (3)$$

We can consider approximating each of the integral terms. For the first integral, we can use the conventional deterministic quadrature:

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$$\int_{\tau_{i-1}}^{\tau_i} f(X(s))ds \approx (\tau_i - \tau_{i-1})f(X_{i-1}) = \delta t f(X_{i-1})$$

And for the second integral, we use the ito formula developed in the previous section:

$$\int_{\tau_{i-1}}^{\tau_i} g(X(s))dW(s) \approx g(X_{i-1})(W(\tau_i) - W(\tau_{i-1}))$$

Combining these together gives us the Euler - Maruyama formula:

$$X_0 = X(0)$$

$$X_i = X_{i-1} + \delta t f(X_{i-1}) + g(X_{i-1})(W(\tau_i) - W(\tau_{i-1})) \quad (4)$$

This method was considered by [13]

Example 1:

$dX_t = \frac{1}{2} a^2 X(t)dt + aX(t)dW(t)$ [18]

$$X_0 = 1, a = 1$$

Solution

$$dX_t = 0.5a^2X(t)dt + aX(t)dW(t) \quad (5)$$

$$\frac{dX(t)}{X(t)} = 0.5a^2dt + adW(t)$$

Integrating,

$$\int_0^t \frac{dX(u)}{X(u)} = 0.5a^2 \int_0^t du + a\int_0^t dW(u)$$

$$\int_0^t \frac{dX(u)}{X(u)} = 0.5a^2t + aW(t) \quad (6)$$

Due to the ito's formula,

$$df(t,X(t)) = \frac{\partial f}{\partial t} f(t,X(t))dt + \frac{\partial f}{\partial x} f(t,X(t))dX(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} f(t,X(t))dX(t)^2$$

We may take $f(t,X(t)) = f(x) = \ln x$ then $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$

$$d(\ln x) = \frac{1}{X(t)}dX(t) + \frac{1}{2}\left[-\frac{1}{X(t)}\right]a^2X^2(t)dt$$

$$dX(t) = f(t, X(t)dt + g(t, X(t))dW(t) \quad (1)$$

$$X(0) = X_0$$

Which can be interpreted as an integral equation:

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If we subtract these equations, then we obtain:

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$$\frac{dX(t)}{X(t)} = 0.5a^2dt + adW(t)$$

Integrating,

$$\int_0^t \frac{dX(u)}{X(u)} = 0.5a^2\int_0^t du + a\int_0^t dW(u)$$

$$\int_0^t \frac{dX(u)}{X(u)} = 0.5a^2t + aW(t) \quad (6)$$

Due to the ito's formula,

$$df(t,X(t)) = \frac{\partial f}{\partial t}f(t,X(t))dt + \frac{\partial f}{\partial x}f(t,X(t))dX(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}f(t,X(t))dX(t)^2$$

We may take $f(t,X(t)) = f(x) = \ln x$ then $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$

$$d(\ln x) = \frac{1}{X(t)}dX(t) + \frac{1}{2}\left[-\frac{1}{X(t)}\right]a^2X^2(t)dt$$

$$= \frac{1}{X(t)} dX(t) - \frac{1}{2} a^2 dt$$

Integrating,

$$\ln X(t) - \ln X(0) = \int_0^t \frac{1}{X(u)} dX(u) - \frac{1}{2} a^2 \int_0^t du$$

$$\ln X(t) - \ln X(0) = \int_0^t \frac{1}{X(u)} dX(u) - \frac{1}{2} a^2 t$$

$$\ln X(t) - \ln X(0) + 0.5a^2 t = \int_0^t \frac{1}{X(u)} dX(u) \qquad (7)$$

Equating equation (6) and (7)

$$0.5a^2 t + aW(t) = \ln X(t) - \ln X(0) + 0.5a^{2}t$$

$$aW(t) = \ln\left(\frac{X(t)}{X(0)}\right)$$

$$\exp\big(aW(t)\big) = \frac{X(t)}{X(0)}$$

$$X(t) = X(0) \exp\big(aW(t)\big)$$

Numerical solution of stochastic differential equations using EMM using step size 2^{-5}

t-value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.792971856820576	0.732852486457998	0.060119370362578
0.1250	0.769452294626993	0.733146674324023	0.036305620302970
0.1875	0.504817088445651	0.459250812386064	0.045566276059587
0.2500	0.350676055681101	0.306088282270949	0.044587773410152
0.3125	0.459166513534967	0.405077067989557	0.054089445545410
0.3750	0.764623033766340	0.650973293833258	0.113649739933082
0.4375	0.571812570252365	0.492986699741621	0.078825870510744
0.5000	0.319501620499993	0.218595746450422	0.100905874049571
0.5625	0.428314888703406	0.294928067471048	0.133386821232358
0.6250	0.385115783719542	0.270839068927022	0.114276714792520

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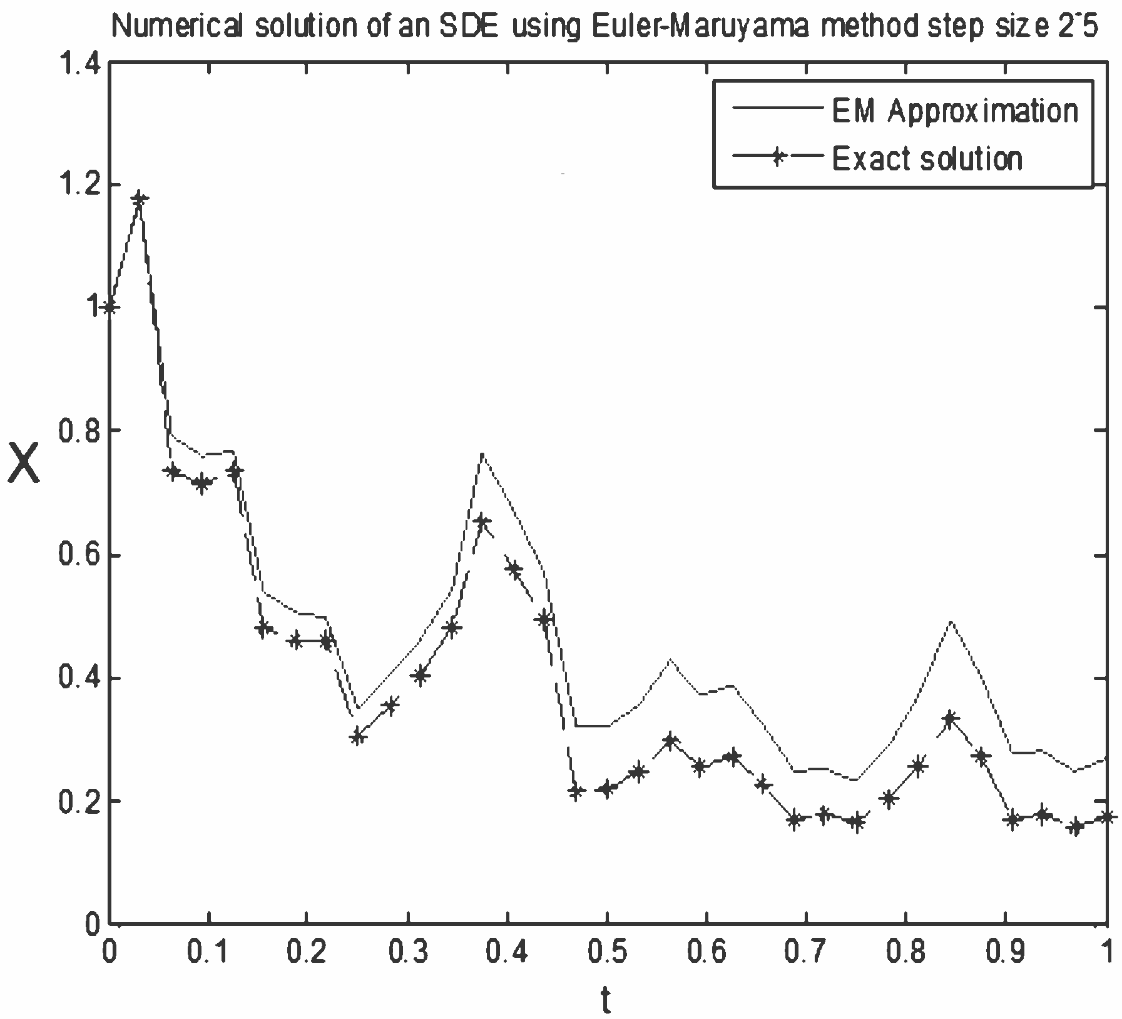

Numerical solution of stochastic differential equations using EMM using step size 2^{-6}

t- value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.830841741986698	0.813918589578289	0.016923152408409
0.1250	0.476650470968802	0.448743537932791	0.027906933036011
0.1875	0.827151096335254	0.768791708978340	0.058359387356914
0.2500	0.446282178258998	0.379350461540254	0.066931716718744
0.3125	0.509294294563069	0.438980935129325	0.070313359433744
0.3750	0.355211406609727	0.307297598010294	0.047913808599433
0.4375	0.527144417093526	0.444995135373477	0.082149281720049
0.5000	0.392832291069872	0.325881011980627	0.066951279089245
0.5625	0.383095209435863	0.303172110127790	0.079923099308073
0.6250	0.476178180718305	0.385047873905375	0.091130306812930

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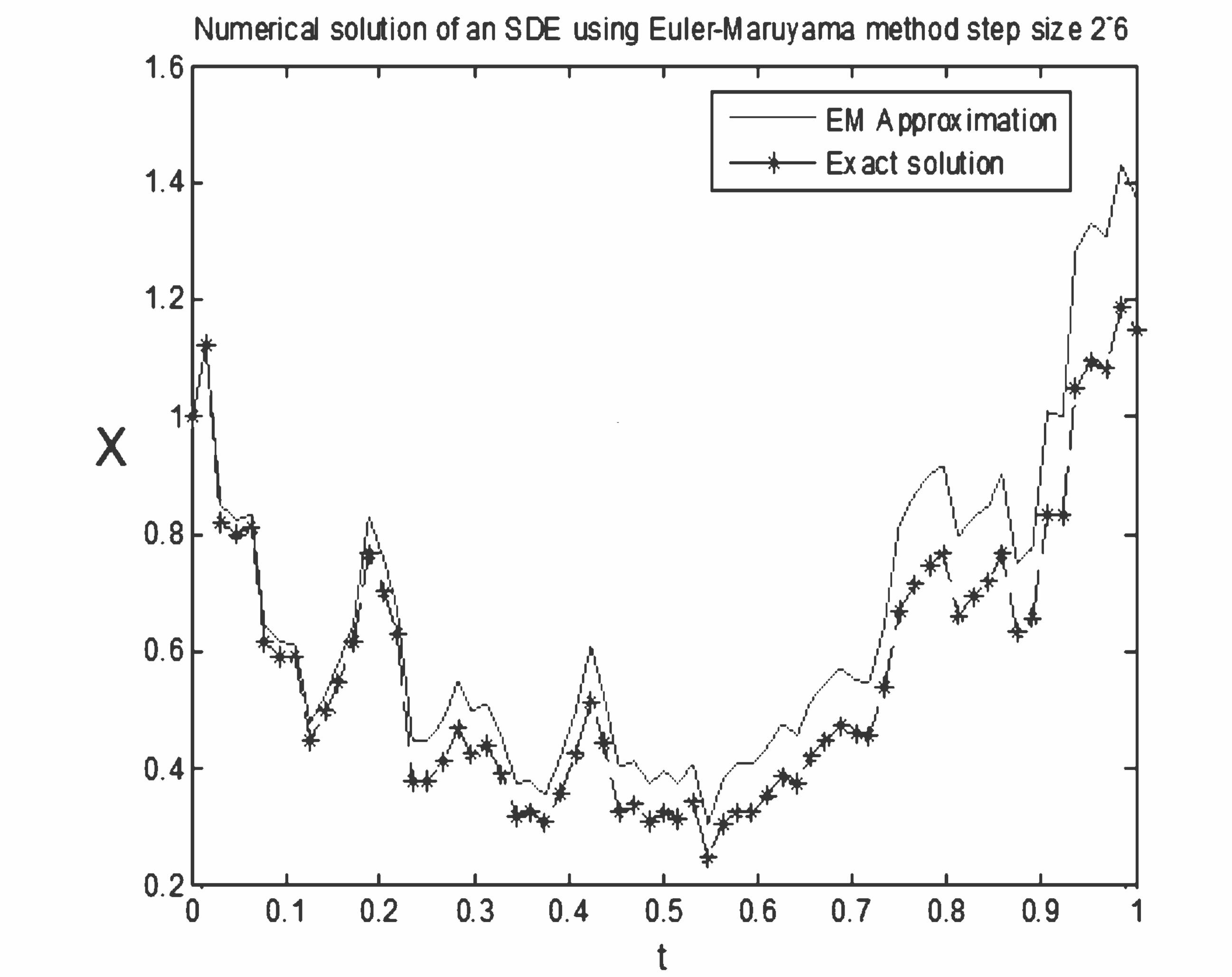

Numerical solution of stochastic differential equations using EMM using step size 2^{-7}

t- value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.592179073997977	0.575911853903834	0.016267220094143
0.1250	0.565244743894177	0.524892144642906	0.040352599251271
0.1875	0.481001290109631	0.450657587248036	0.030343702861595
0.2500	0.516489081614384	0.474216458466448	0.042272623147936
0.3125	0.591764110348729	0.537056194540920	0.054707915807809
0.3750	0.866560586851293	0.790705689598516	0.075854897252777
0.4375	0.816997127132094	0.756002903350736	0.060994223781358
0.5000	1.251000232820100	1.154387971952310	0.096612260867790
0.5625	1.653223753057290	1.543604143011460	0.109619610045830
0.6250	1.742340234153280	1.616910605389530	0.125429628763750

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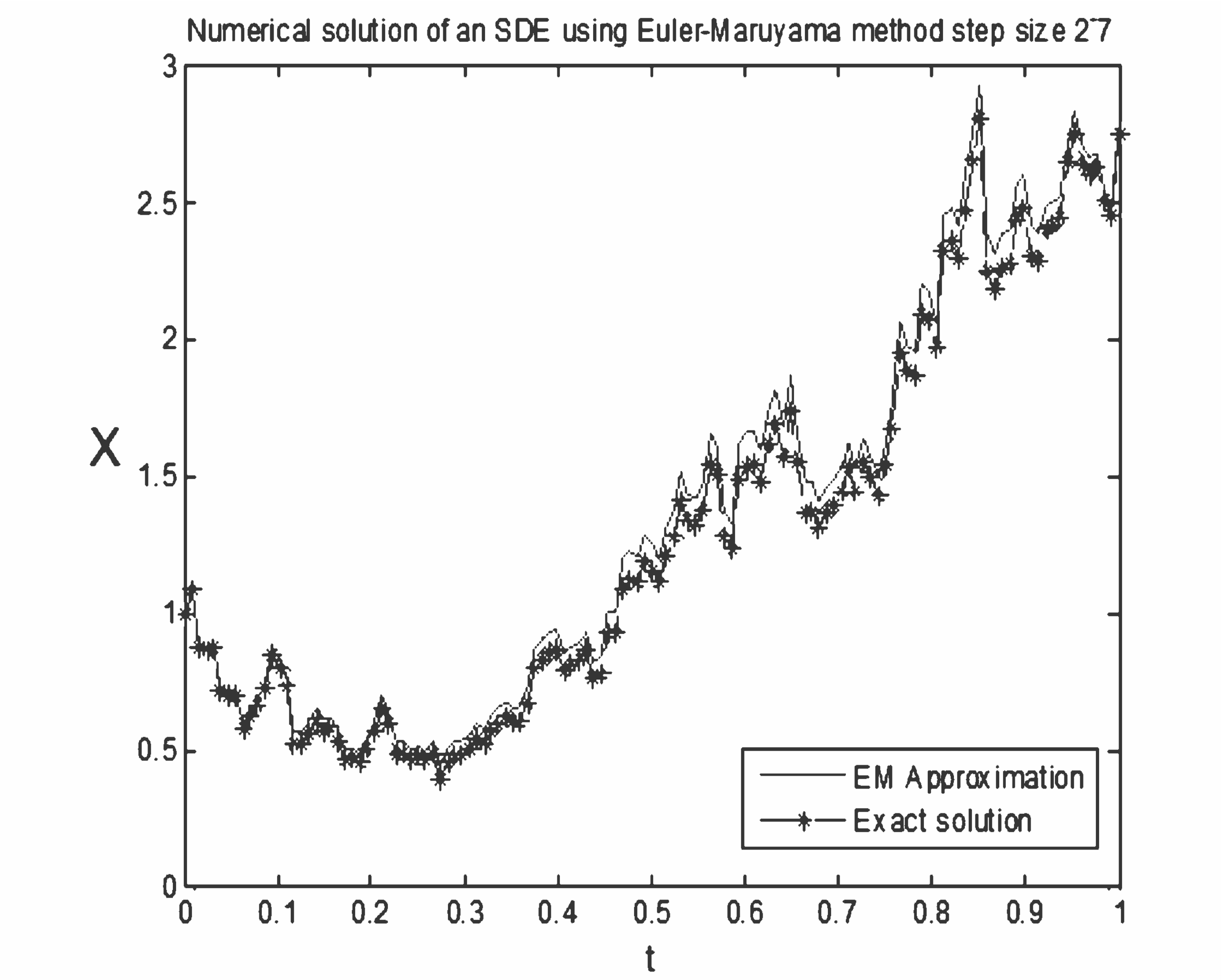

Numerical solution of stochastic differential equations using EMM using step size 2^{-8}

t- value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.668043545181747	0.645124935848592	0.022918609333155
0.1250	0.626763345346449	0.602001762853705	0.024761582492744
0.1875	0.903685269923684	0.865515434179290	0.038169835744394
0.2500	1.171578411371640	1.128229284251590	0.043349127120050
0.3125	1.480835288132010	1.429881530073540	0.050953758058470
0.3750	1.401052699132930	1.371002029842480	0.030050669290450
0.4375	1.848715225905090	1.802135635474780	0.046579590430310
0.5000	2.065901822124680	2.057382075656720	0.008519746467960
0.5625	2.556748627063660	2.545988428126860	0.010760198936800
0.6250	1.985530788090960	2.010389557659420	0.024858769568460

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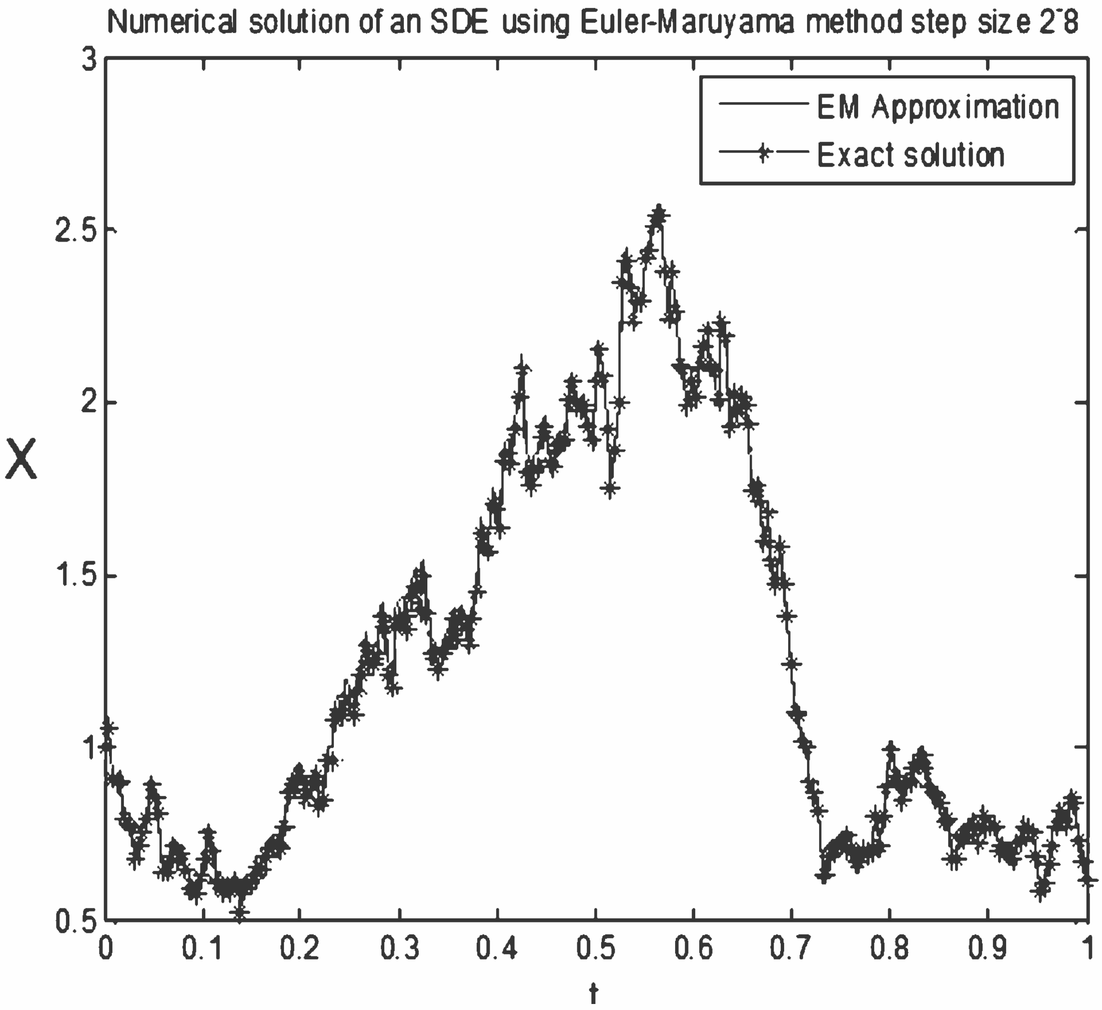
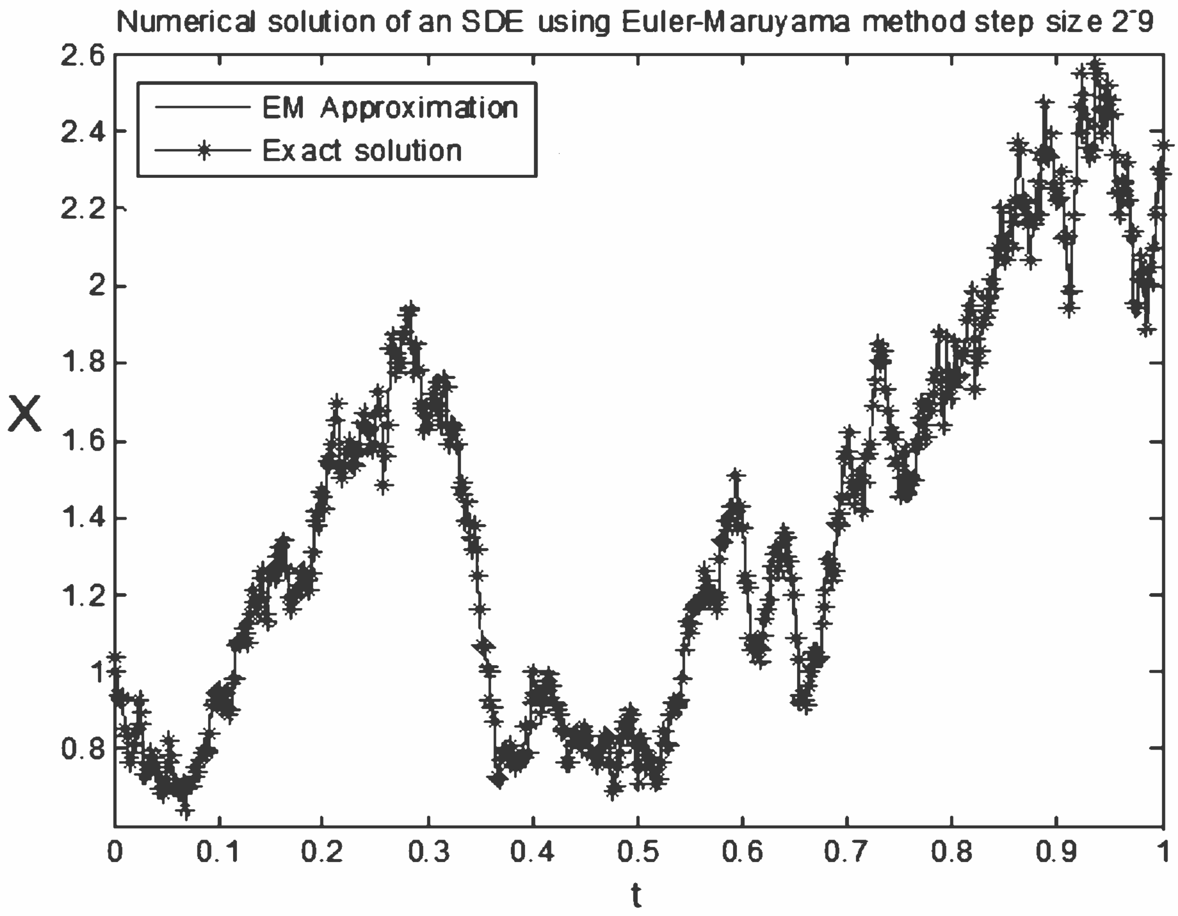


Numerical solution of stochastic differential equations using EMM using step size 2^{-9}

t- value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.718671748167676	0.704862968745263	0.013808779422413
0.1250	1.118481217017121	1.098434761853992	0.020046455163129
0.1875	1.269284421139427	1.256508593734478	0.012775827404949
0.2500	1.670383497022962	1.668209597495599	0.002173899527363
0.3125	1.624166625283062	1.635294503526769	0.011127878243707
0.3750	0.810920442543822	0.804591909013427	0.006328533530395
0.4375	0.823183887302603	0.814244700163518	0.008939187139085
0.5000	0.723620556846645	0.713603856690012	0.010016700156633
0.5625	1.238537625370480	1.216641719164480	0.021895906206000
0.6250	1.208806738431730	1.190297627759960	0.018509110671770



Question 2:

$$dX(t) = -a^2X(t)\big(1 - X^2(t)\big)dt + a\big(1 - X^2(t)\big)dW(t)$$

$$a = 0.25\,, X_0 = 0.65$$

The problem was solved by [1], [7] with constant $a = 0.5\,, X_0 = 0$, [18].

Solution:

$$\frac{dX(t)}{1-X^2(t)} = -a^2X(t)dt + adW(t) \qquad (9)$$

$$\int_0^t \frac{dX(u)}{1 - X^2(u)} = -a^2\int_0^t X(u)du + a\int_0^t dW(u)$$

$$\int_0^t \frac{dX(u)}{1-X^2(u)} = -a^2X(t) + aW(t) \qquad (10)$$

Due to the ito's formula,

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$$df(t,X(t)) = \frac{\partial f}{\partial t} f(t,X(t))dt + \frac{\partial f}{\partial x} f(t,X(t))dX(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} f(t,X(t))dX(t)^2$$

We may take $f(t,X(t)) = f(x) = tanh^{-1}X(t)$, then $f'(x) = \frac{1}{1-X^2}$, $f''(x) = \frac{2X}{(1-X^2)^2}$

Then,

$$d(tanh^{-1}X(t)) = \frac{1}{1-X^2(t)} dX(t) + \frac{1}{2}\left[\frac{2X(t)}{(1-X^2(t))^2}\right] dX(t)^2$$

$$= \frac{1}{1-X^2(t)} dX(t) + \frac{X(t)}{(1-X^2)^2}[a(1-X^2)^2 dt]$$

$$= \frac{1}{1-X^2(t)} dX(t) + X(t)(a)dt$$

Integrating,

$$tanh^{-1}X(t) - tanh^{-1}X(0) = \int_0^t \frac{1}{1-X^2(u)} dX(u) + 0.25X(t) \qquad (11)$$

$$tanh^{-1}X(t) - tanh^{-1}X(0) - 0.25X(t) = \int_0^t \frac{1}{1-X^2(u)} dX(u)$$

Combining equations (10) and (11)

$$tanh^{-1}X(t) - tanh^{-1}X(0) - 0.25X(t) = \int_0^t \frac{1}{1-X^2(u)} dX(u) = -0.25X(t) + 0.5W(t)$$

$$tanh^{-1}X(t) - tanh^{-1}X(0) - 0.25X(t) = -0.25X(t) + 0.5W(t)$$

$$tanh^{-1}X(t) - tanh^{-1}X(0) = 0.5W(t)$$

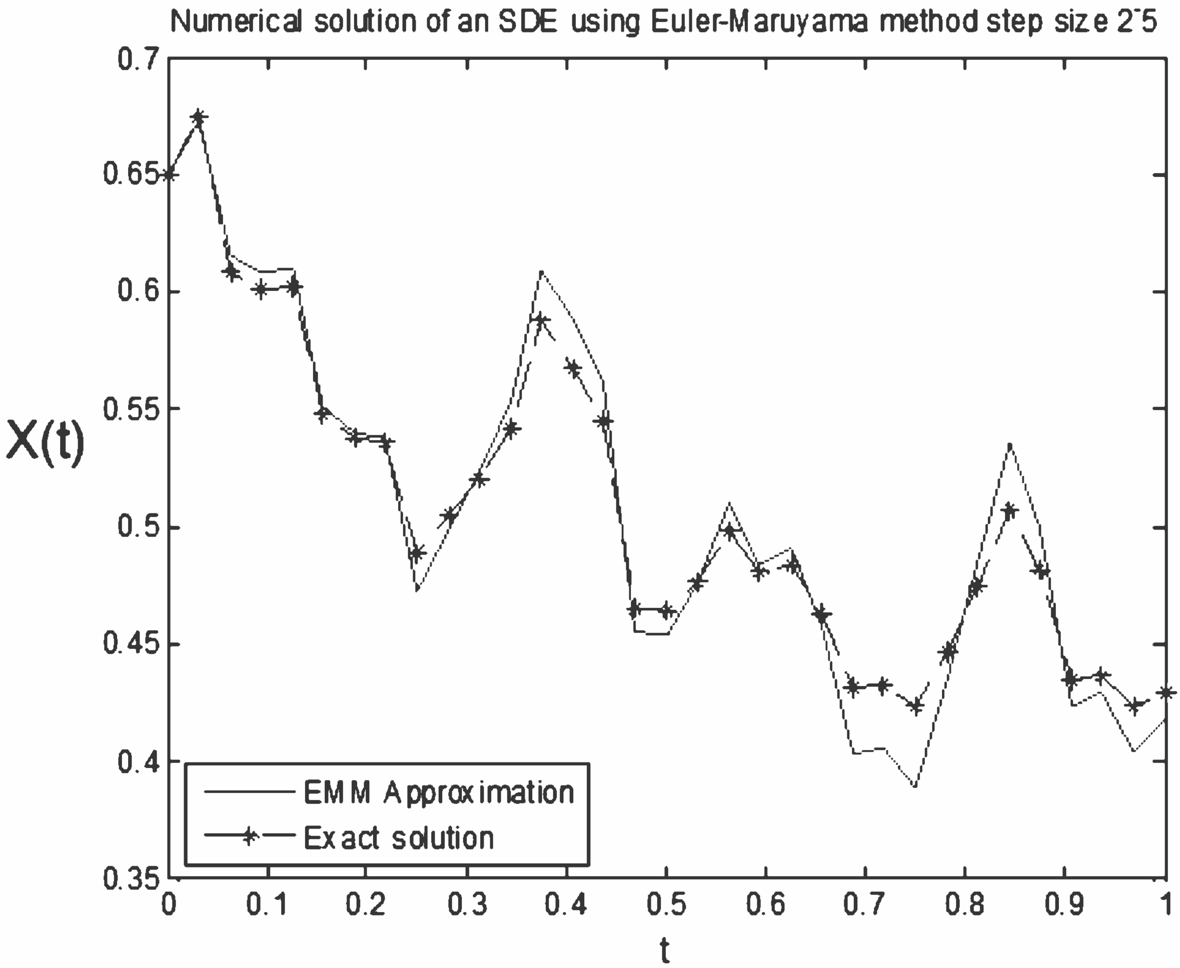
$$tanh^{-1}X(t) = 0.5W(t) + tanh^{-1}X(0)$$

$$X(t) = tanh[0.5W(t) + tanh^{-1}X(0)]$$

Numerical solution of stochastic differential equations using EMM using step size 2^{-5}

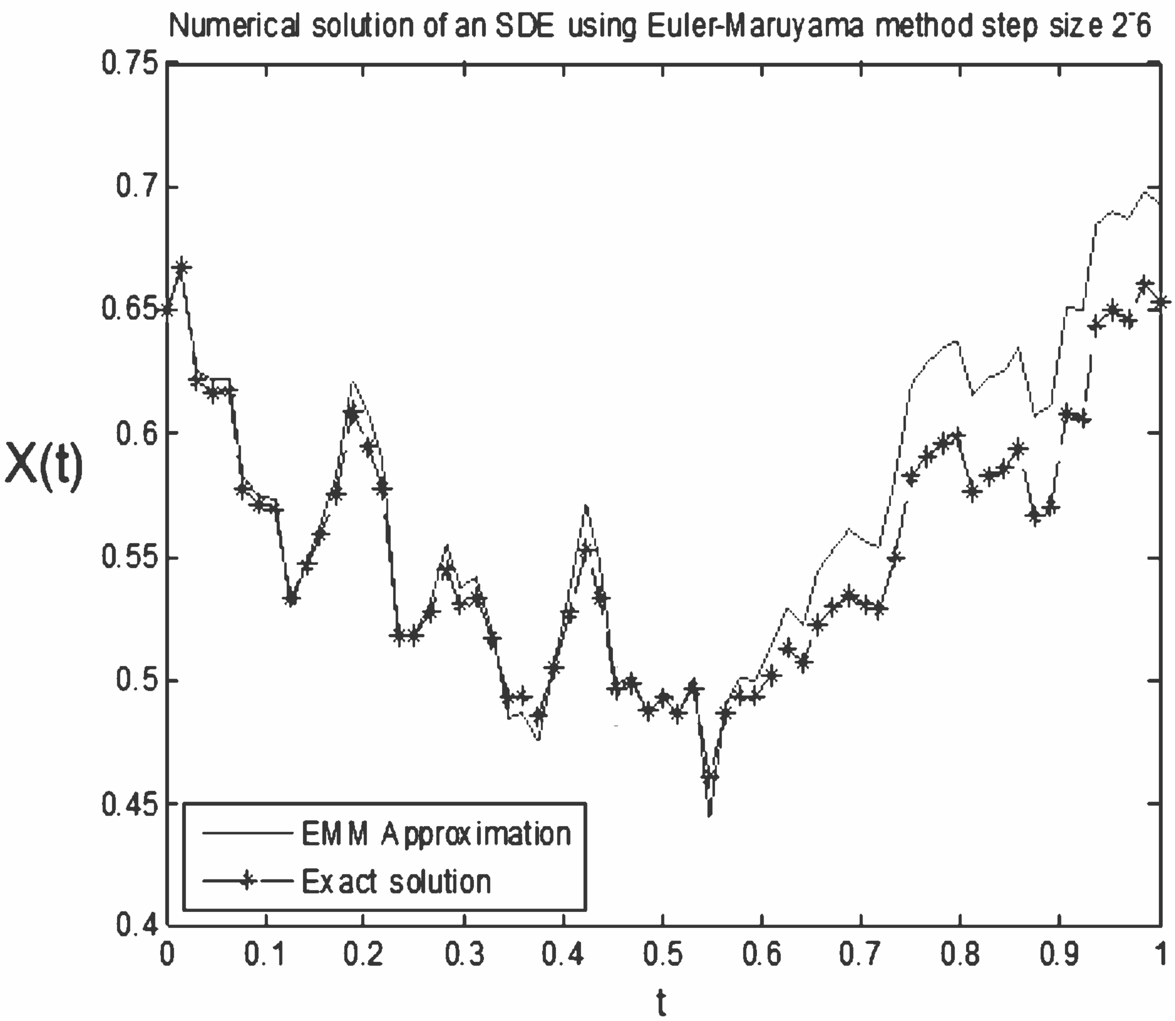
t-value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.615238312843228	0.608109061861152	0.007129250982076
0.1250	0.610538623128736	0.602082356050494	0.008456267078242
0.1875	0.540179430271432	0.538223101477412	0.001956328794020
0.2500	0.472532297247553	0.488292209906610	0.015759912659057
0.3125	0.523183544189210	0.520772103428445	0.002411440760765
0.3750	0.609550392860017	0.587892422046201	0.021657970813816
0.4375	0.561870675494029	0.544677014565642	0.017193660928387

0.5000	0.454256491476679	0.464522020207296	0.010265528730617
0.5625	0.510439943252099	0.498256064349205	0.012183878902894
0.6250	0.490520779248640	0.483912001255832	0.006608777992808



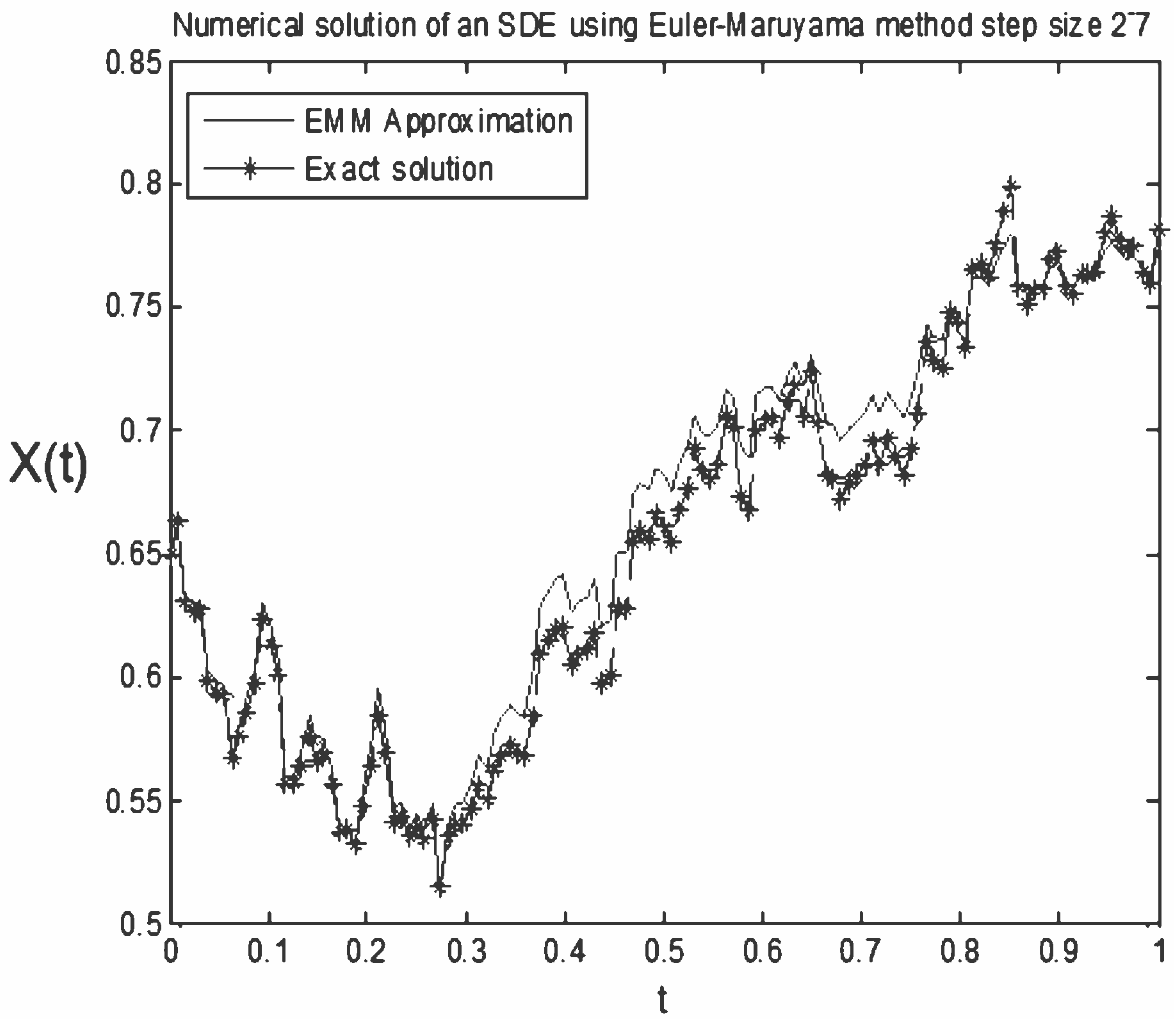
Numerical solution of stochastic differential equations using EMM using step size 2^{-6}

t-value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.622434630313017	0.617180212162280	0.005254418150737
0.1250	0.529935702850741	0.533647241161016	0.003711538310275
0.1875	0.621752369276110	0.609537959095290	0.012214410180820
0.2500	0.517996578796898	0.517930548307746	0.000066030489152
0.3125	0.541740907696983	0.533658698938304	0.008082208758679
0.3750	0.475023745337134	0.485842682445806	0.010818937108672
0.4375	0.547797116183121	0.533296891497423	0.014500224685698
0.5000	0.494277942696807	0.493119353851824	0.001158588844983
0.5625	0.489521447187964	0.486842594254703	0.002678852933261
0.6250	0.529757443757840	0.512804276759769	0.016953166998071



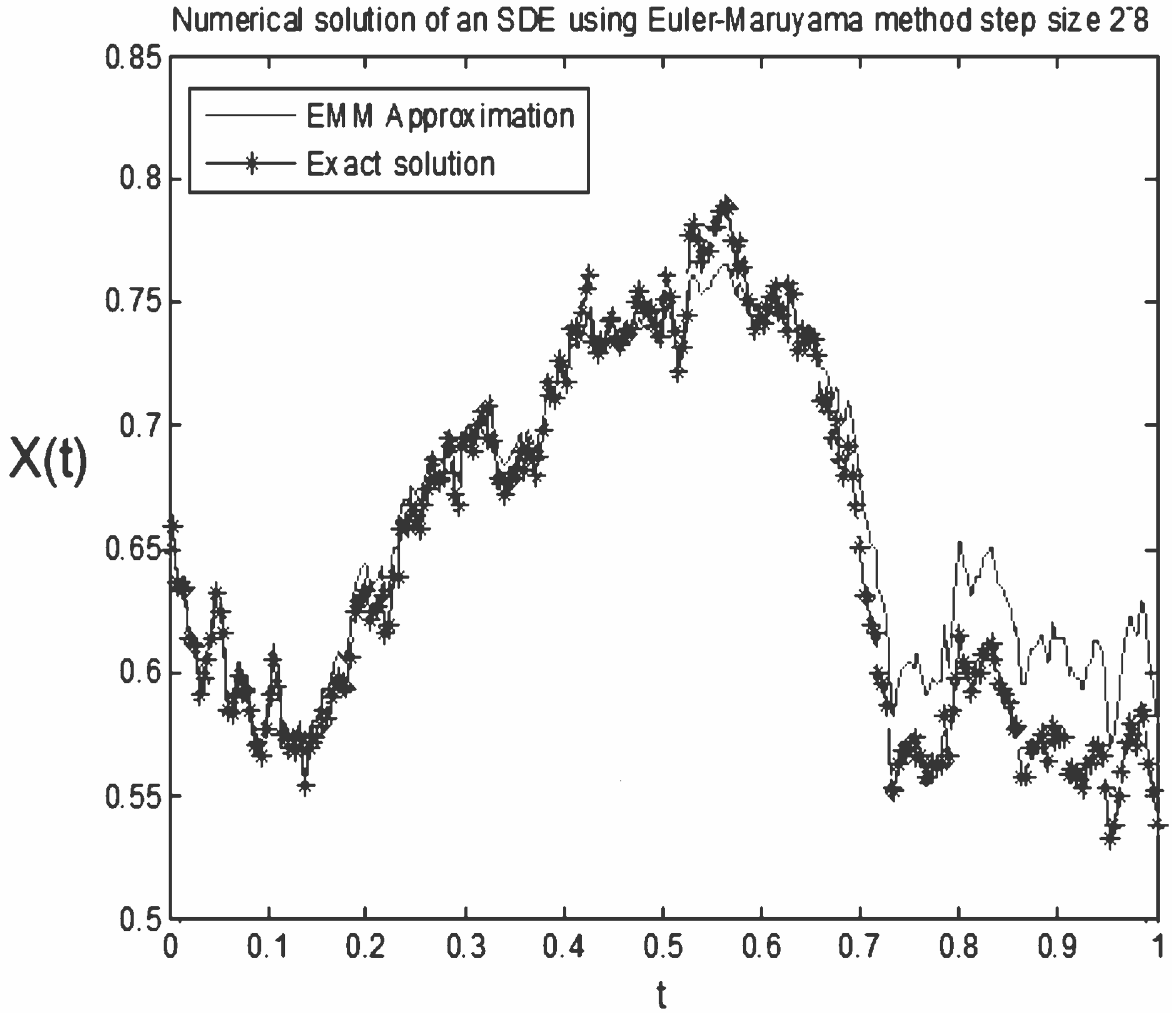
Numerical solution of stochastic differential equations using EMM using step size 2^{-7}

t-value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.567828497526464	0.566798783968530	0.001029713557934
0.1250	0.559891129276148	0.556463803006885	0.003427326269263
0.1875	0.531567415256637	0.532544887367346	0.000977472110709
0.2500	0.544213974168055	0.539239345686061	0.004974628481994
0.3125	0.567709743837613	0.555305073126635	0.012404670710978
0.3750	0.628838681883073	0.608538510623765	0.020300171259308
0.4375	0.619854508300622	0.597582259370807	0.022272248929815
0.5000	0.681165151248706	0.661661408471524	0.019503742777182
0.5625	0.716774956996478	0.706719583442207	0.010055373554271
0.6250	0.723097174279630	0.712416008986527	0.010681165293103



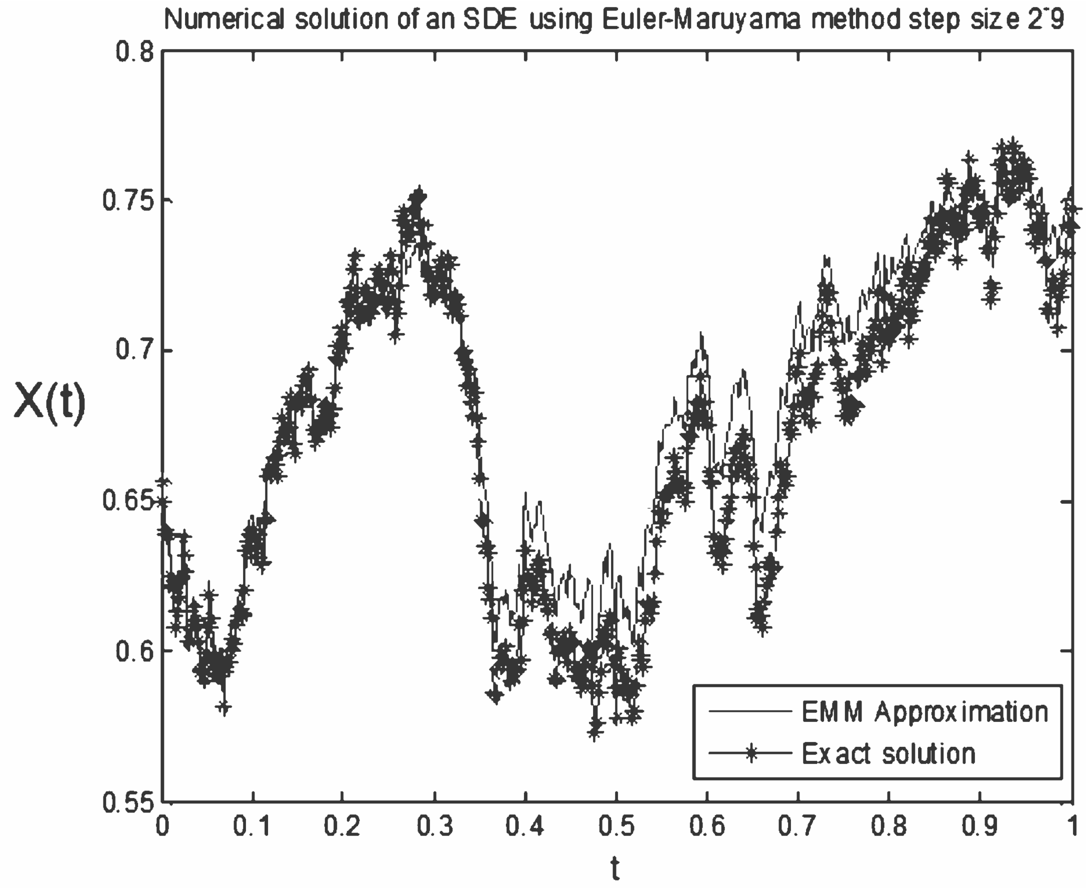
Numerical solution of stochastic differential equations using EMM using step size 2^{-8}

t-value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.587898306196455	0.583904126610705	0.003994179585750
0.1250	0.577365732605268	0.572058441068058	0.005307291537210
0.1875	0.635137123861844	0.624131767392211	0.011005356469633
0.2500	0.672277179925475	0.663258453974430	0.009018725951045
0.3125	0.703122892986596	0.700140818338637	0.002982074647959
0.3750	0.696053936077163	0.687891746091963	0.008162189985200
0.4375	0.730090638794333	0.733584091776565	0.003493452982232
0.5000	0.742796982140519	0.751602398119085	0.008805415978566
0.5625	0.765754904971199	0.788818331968186	0.023063426996987
0.6250	0.738317390334205	0.737614273182266	0.000703117151939



Numerical solution of stochastic differential equations using EMM using step size 2^{-9}

t-value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.599720812990287	0.595187408698017	0.004533404292270
0.1250	0.665872954832284	0.661935339945974	0.003937614886310
0.1875	0.683104724562316	0.680404358684769	0.002700365877547
0.2500	0.718027856294013	0.725717773140614	0.007689916846601
0.3125	0.714612585851372	0.717534114268776	0.002921528417404
0.3750	0.618703881777585	0.599893382123145	0.018810499654440
0.4375	0.621014517645261	0.599422470689938	0.021592046955323
0.5000	0.600818249121088	0.577830095516106	0.022988153604982
0.5625	0.679821210390718	0.657870459363592	0.021950751027126
0.6250	0.676540623493794	0.651048281915320	0.025492341578474



Milstein's Method

The Milstein method has order1, meaning that it will converge to the correct stochastic solution process faster than Euler-Maruyama method as the step size goes to 0. The Milstein method is identical to the Euler-Maruyama method if there is no X term in the diffusion part $b(X,t)$ of the equation.

We now attempt to construct the Milstein's method. In order to do this, we will first derive the stochastic chain rule, using Ito's calculus [11],[16].

Ito's Lemma

Derivation

$$\delta G \approx \frac{dG}{dx}\delta x \tag{12}$$

If more accuracy is required, a Taylor series expansion of δG can be used:

$$\delta G = \frac{dG}{dx}\delta x + \frac{1}{2}\frac{d^2G}{dx^2}\delta x^2 + \frac{1}{6}\frac{d^3G}{dx^3}\delta x^3 + \cdots$$

For a continuous and differentiable function G of two variables x and y, the result analogus to equation (12) is

$$\delta G \approx \frac{dG}{dx}\delta x + \frac{dG}{dy}\delta y \tag{13}$$

And the Taylor series expansion of δG is

$$\delta G = \frac{\delta G}{\delta x}\delta x + \frac{\delta G}{\delta y}\delta y + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}\delta x^2 + \frac{\delta^2 G}{\delta x \delta y}\delta x \delta y + \frac{1}{2}\frac{\delta^2 G}{\delta y^2}\delta y^2 + \cdots \tag{14}$$

In the limit as δx and δy tends to zero, equation (14) becomes

$dG = \frac{\delta G}{\delta x}dx + \frac{\delta G}{\delta y}dy$ (15)

Now we extend equation (15) to cover functions of variables following ito process. Suppose that a variable x follows the ito process in equation (15), that is,

$dx = a(x,t)dt + b(x,t)dW$ (16)

And that G is some function of x and of time t.

$\delta G = \frac{\delta G}{\delta x}\delta x + \frac{\delta G}{\delta t}\delta t + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}\delta x^2 + \frac{\delta^2 G}{\delta x \delta t}\delta x \delta t + \frac{1}{2}\frac{\delta^2 G}{\delta t^2}\delta t^2 + \cdots$ (17)

Equation (16) can be discretized to

$$\delta x = a(x,t)\delta t + b(x,t)\epsilon\sqrt{\delta t}$$

Or, if arguments are dropped,

$\delta x = a\delta t + b\epsilon\sqrt{\delta t}$ (18)

From (18), we have

$\delta x^2 = b^2\epsilon^2\delta t + terms\ of\ higher\ order\ in\ \delta t.$ (19)

Making use of equation (19), equation (17) becomes

$dG = \frac{\delta G}{\delta x}dx + \frac{\delta G}{\delta t}dt + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}b^2 dt$ (20)

This is ito's lemma. If we substitute for dx from equation (16), then equation (20) becomes

$dG = \left(\frac{\delta G}{\delta x}a + \frac{\delta G}{\delta t} + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}b^2\right)dt + \frac{\delta G}{\delta x}b\,dW$ (21)

Which becomes

$dG = \left(\frac{dG}{dX}a + \frac{1}{2}b^2\frac{d^2 G}{dX^2}\right)dt + b\frac{dG}{dX}dW$ (22)

Derivation of Milstein's Method

Setting successively $t = s$ and $t = \tau_{j-1}$ in the equivalent integral form of (22) and subtracting gives:

$$G(X(s)) = G\big(X(\tau_{j-1})\big) + \int_{\tau_{j-1}}^{s}\left\{a(X(\tau))\frac{dG}{dX}(X(\tau)) + \frac{1}{2}b(X(\tau))^2\frac{d^2G}{dX^2}(X(\tau))\right\}d\tau + \int_{\tau_{j-1}}^{s}\left\{b\big(X(\tau)\big)\frac{dG}{dX}\big(X(\tau)\big)\right\}dW(\tau)$$

Or

$$G(X(s)) = G\big(X(\tau_{j-1})\big) + \int_{\tau_{j-1}}^{s} L^0 G(X(\tau))d\tau + \int_{\tau_{j-1}}^{s} L^1 G\big(X(\tau)\big)dW(\tau)$$

Where:

$$L^0 = a(X(\tau))\frac{d}{dX} + \frac{1}{2}b(X(\tau))^2\frac{d^2}{dX^2}$$

And $L^1 = b\big(X(\tau)\big)\frac{d}{dX}$

Taking in turn $G = a$ and $G = b$ gives

$$a(X(s)) = a\big(X(\tau_{j-1})\big) + \int_{\tau_{j-1}}^{s} L^0\, a(X(\tau))d\tau + \int_{\tau_{j-1}}^{s} L^1\, a\big(X(\tau)\big)dW(\tau)$$

$$b(X(s)) = b\big(X(\tau_{j-1})\big) + \int_{\tau_{j-1}}^{s} L^0\, b(X(\tau))d\tau + \int_{\tau_{j-1}}^{s} L^1\, b\big(X(\tau)\big)dW(\tau)$$

Substituting these equations into the integral form of the autonomous SDE

$X(\tau_j) = X(\tau_{j-1}) + \int_{\tau_{j-1}}^{\tau_j} a(X(s))ds + \int_{\tau_{j-1}}^{\tau_j} a\big(X(s)\big)dW(s)$ gives

$$X(\tau_j) = X(\tau_{j-1}) + \int_{\tau_{j-1}}^{\tau_j} \left\{a\big(X(\tau_{j-1})\big) + \int_{\tau_{j-1}}^{s} L^0\, a(X(\tau))d\tau + \int_{\tau_{j-1}}^{s} L^1\, a\big(X(\tau)\big)dW(\tau)\right\} ds$$
$$+ \int_{\tau_{j-1}}^{\tau_j} \left\{b\left(X(\tau_{j-1})\right) + \int_{\tau_{j-1}}^{s} L^0\, b\big(X(\tau)\big)d\tau + \int_{\tau_{j-1}}^{s} L^1\, b\big(X(\tau)\big)dW(\tau)\right\} dW(\tau)$$

Which can be re-written as:

$$X(\tau_j) = X(\tau_{j-1}) + \delta t a\big(X(\tau_{j-1})\big) + b\big(X(\tau_{j-1})\big)\int_{\tau_{j-1}}^{\tau_j} dW(s) + \int_{\tau_{j-1}}^{\tau_j}\int_{\tau_{j-1}}^{s} L^0 a\big(X(\tau)\big)d\tau ds$$
$$+ \int_{\tau_{j-1}}^{\tau_j}\int_{\tau_{j-1}}^{s} L^1 a\big(X(\tau)\big)dW(\tau)ds + \int_{\tau_{j-1}}^{\tau_j}\int_{\tau_{j-1}}^{s} L^0 b\big(X(\tau)\big)d\tau dW(s)$$
$$+ \int_{\tau_{j-1}}^{\tau_j}\int_{\tau_{j-1}}^{s} L^1 b\big(X(\tau)\big)dW(\tau)dW(s)$$

Note that the first line of this formula is the Euler-Maruyama method. To obtain Milstein's method, we expand the $L^1 b\big(X(\tau)\big)$ term using the definition of the operator L^1 and the formula for $b(X(s))$, above to obtain:

$$L^1 b\big(X(\tau)\big) = L^1 b\big(X(\tau_{j-1})\big) + \int_{\tau_{j-1}}^{s} L^0 L^1 b\big(X(m)\big)dm + \int_{\tau_{j-1}}^{s} L^1 L^1 b\big(X(m)\big)dW(m)$$

Substituting this into the previous equation and rearranging slightly gives:

$$X(\tau_j) = X(\tau_{j-1}) + \delta t a\big(X(\tau_{j-1})\big) + b\big(X(\tau_{j-1})\big)\int_{\tau_{j-1}}^{\tau_j} dW(s)$$
$$+ \int_{\tau_{j-1}}^{\tau_j}\int_{\tau_{j-1}}^{s} L^1 b\left(X(\tau_{j-1})\right) dW(\tau)dW(s) + \int_{\tau_{j-1}}^{\tau_j}\int_{\tau_{j-1}}^{s} L^0 a\big(X(\tau)\big)d\tau ds$$
$$+ \int_{\tau_{j-1}}^{\tau_j}\int_{\tau_{j-1}}^{s} L^1 a\big(X(\tau)\big)dW(\tau)ds + \int_{\tau_{j-1}}^{\tau_j}\int_{\tau_{j-1}}^{s} L^0 b\big(X(\tau)\big)d\tau dW(s)$$

$$+\int_{\tau_{j-1}}^{\tau_j}\int_{\tau_{j-1}}^{s}\int_{\tau_{j-1}}^{\tau} L^0L^1 b\big(X(m)\big)dm\,dW(\tau)dW(s)$$
$$+\int_{\tau_{j-1}}^{\tau_j}\int_{\tau_{j-1}}^{s}\int_{\tau_{j-1}}^{\tau} L^1L^1 b\big(X(m)\big)dW(m)dW(\tau)dW(s)$$

Milstein's method follows from the first two lines of the above expression, which can be written as:

$$X(\tau_j) = X(\tau_{j-1}) + \delta t a\big(X(\tau_{j-1})\big) + b\big(X(\tau_{j-1})\big)\int_{\tau_{j-1}}^{\tau_j} dW(s)$$
$$+ b\Big(X(\tau_{j-1})\Big)\, b'\Big(X(\tau_{j-1})\Big)\int_{\tau_{j-1}}^{\tau_j}\int_{\tau_{j-1}}^{s} dW(\tau)dW(s)$$

Finally, Milstein's higher order method is:

$$X_j = X_{j-1} + \delta t a(X_{j-1}) + b(X_{j-1})\Big(W(\tau_j) - W(\tau_{j-1})\Big) + \tfrac{1}{2} b(X_{j-1}) b'(X_{j-1})\Big(\big(W(\tau_j) - W(\tau_{j-1})\big)^2 - \delta t\Big) \qquad (23)$$

For $j = 1,2,\cdots,L$ with $X_0 = X(0)$

The method in (23) was considered by [5] and [13].

Numerical solution of stochastic differential equations using Milstein method MM 2^{-5}

t-value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.792971856820576	0.803243469643136	0.010271612822560
0.1250	0.769452294626993	0.779429140117746	0.009976845490753
0.1875	0.504817088445651	0.516472578511676	0.011655490066025
0.2500	0.350676055681101	0.362112722071500	0.011436666390399
0.3125	0.459166513534967	0.473790548017275	0.014624034482308
0.3750	0.764623033766340	0.784370784495660	0.019747750729320
0.4375	0.571812570252365	0.587269112593059	0.015456542340694
0.5000	0.319501620499993	0.344405887420135	0.024904266920142
0.5625	0.428314888703406	0.461188320008260	0.032873431304854
0.6250	0.385115783719542	0.414894110966598	0.029778327247056

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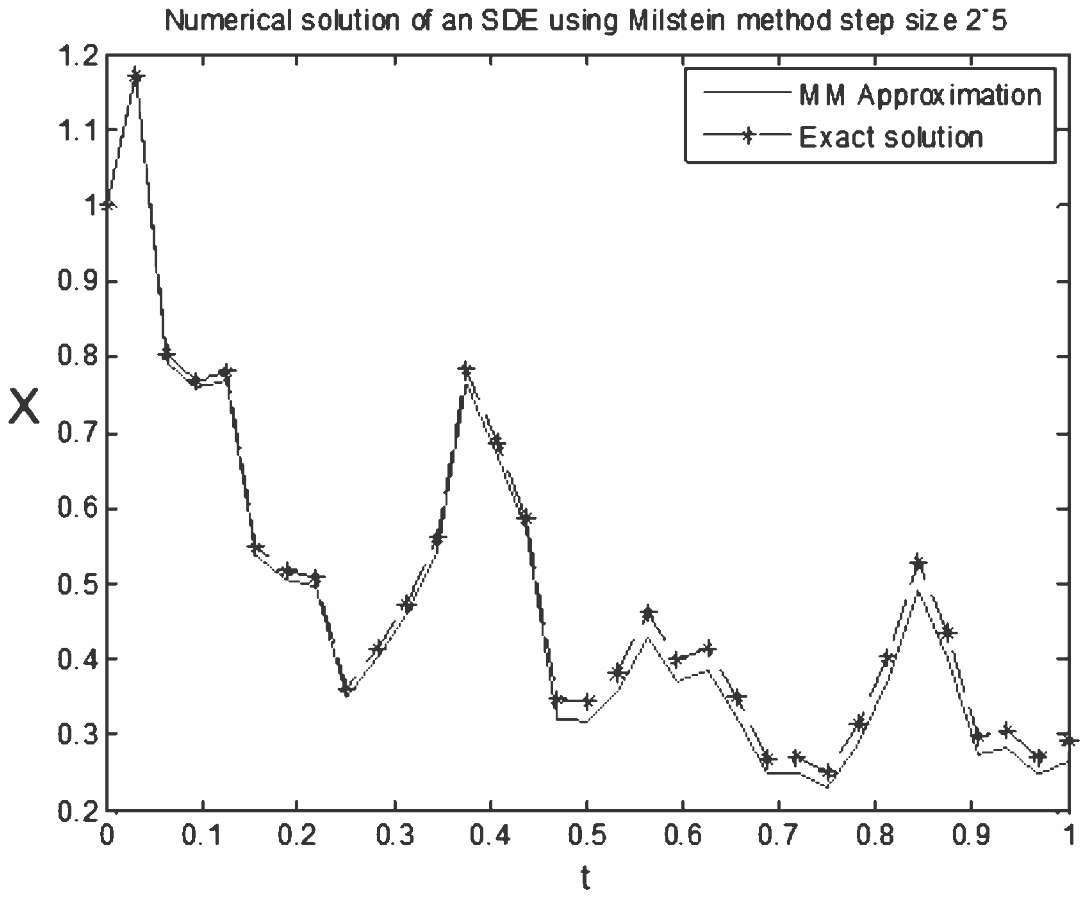

Numerical solution of stochastic differential equations using Milstein method MM 2^{-6}

t-value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.830841741986698	0.834311075973400	0.003469333986702
0.1250	0.476650470968802	0.481664870068719	0.005014399099917
0.1875	0.827151096335254	0.833778580600230	0.006627484264976
0.2500	0.446282178258998	0.456967381947110	0.010685203688112
0.3125	0.509294294563069	0.521370579606078	0.012076285043009
0.3750	0.355211406609727	0.364249517234915	0.009038110625188
0.4375	0.527144417093526	0.539469466136211	0.012325049042685
0.5000	0.392832291069872	0.403757511051427	0.010925219981555
0.5625	0.383095209435863	0.395024084033319	0.011928874597456
0.6250	0.476178180718305	0.490905585945087	0.014727405226782

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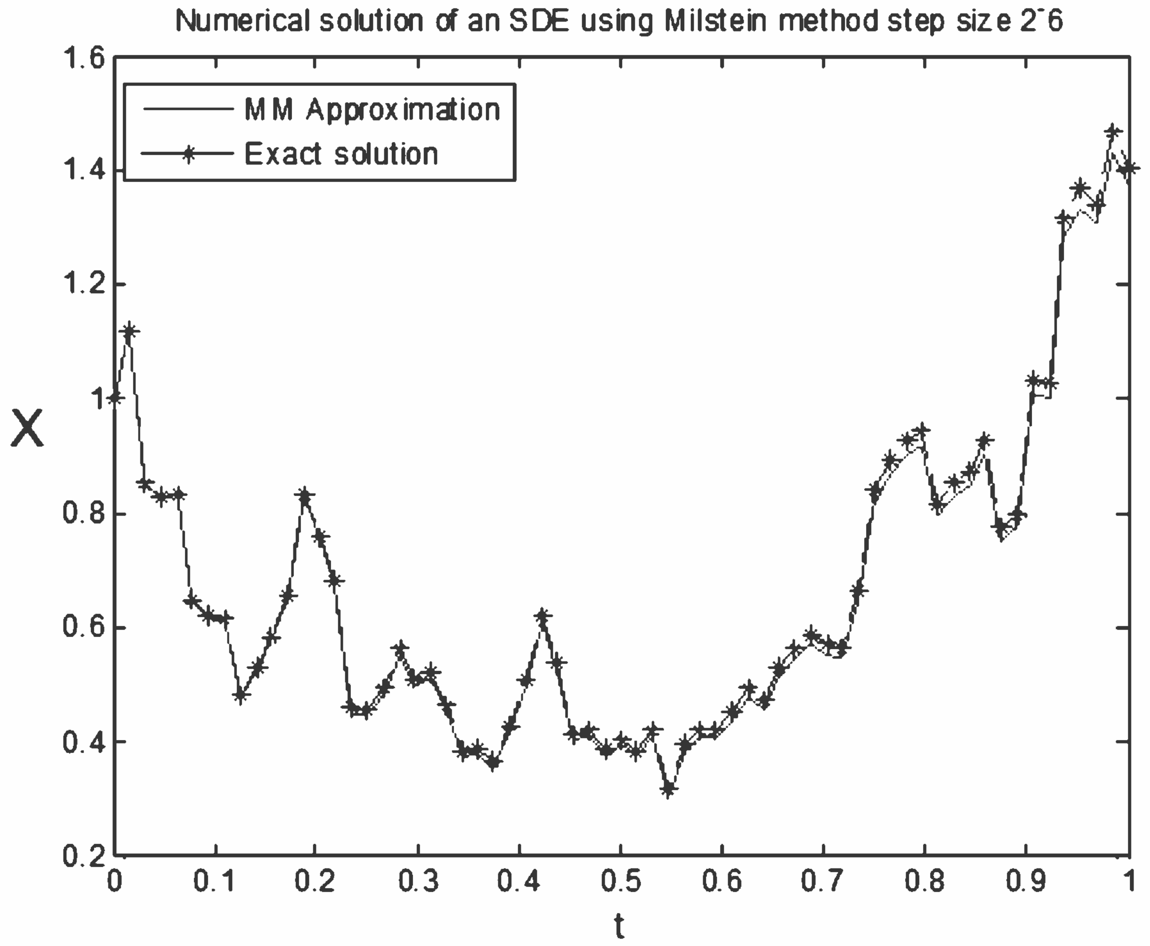

Numerical solution of stochastic differential equations using Milstein method MM 2^{-7}

t-value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.592179073997977	0.594249124604452	0.002070050606475
0.1250	0.565244743894177	0.569594338405651	0.004349594511474
0.1875	0.481001290109631	0.484940376000521	0.003939085890890
0.2500	0.516489081614384	0.521076818472843	0.004587736858459
0.3125	0.591764110348729	0.597577080007985	0.005812969659256
0.3750	0.866560586851293	0.874154389418787	0.007593802567494
0.4375	0.816997127132094	0.824611918175777	0.007614791043683
0.5000	1.251000232820100	1.260403926693859	0.009403693873759
0.5625	1.653223753057290	1.664848083102314	0.011624330045024
0.6250	1.742340234153280	1.753553622344225	0.011213388190945

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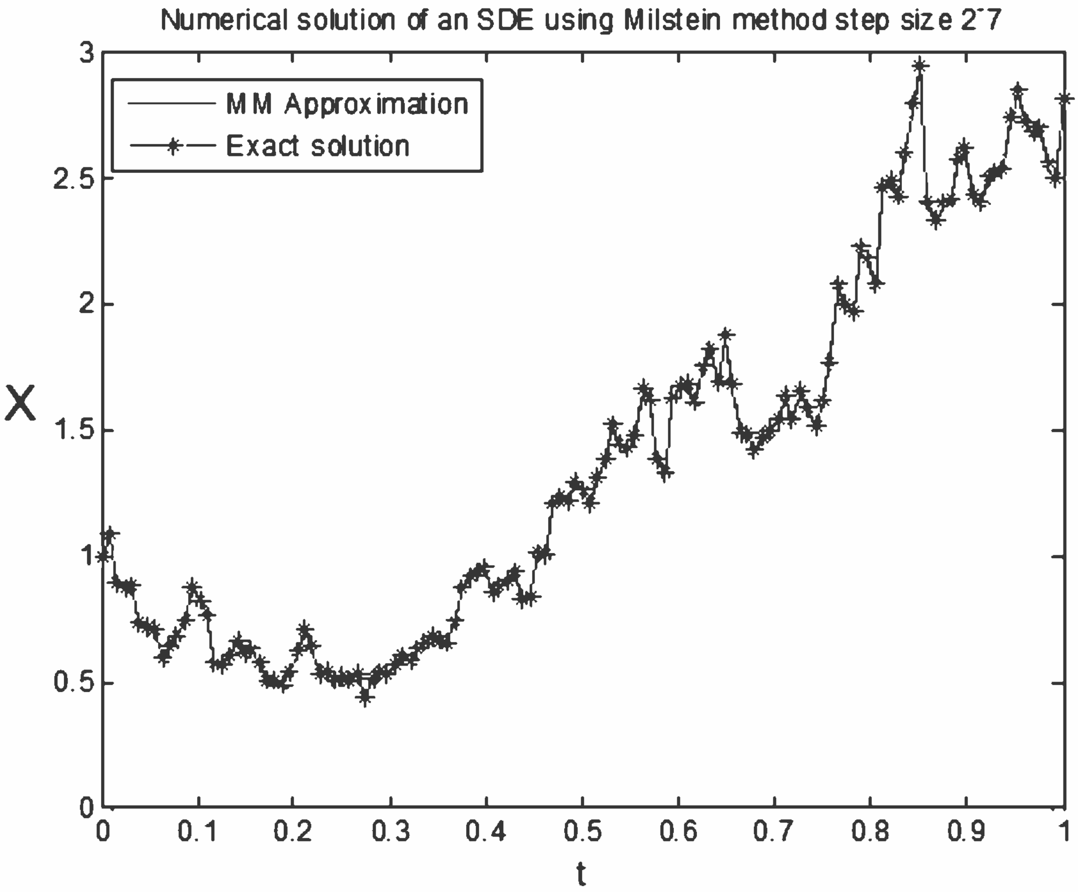
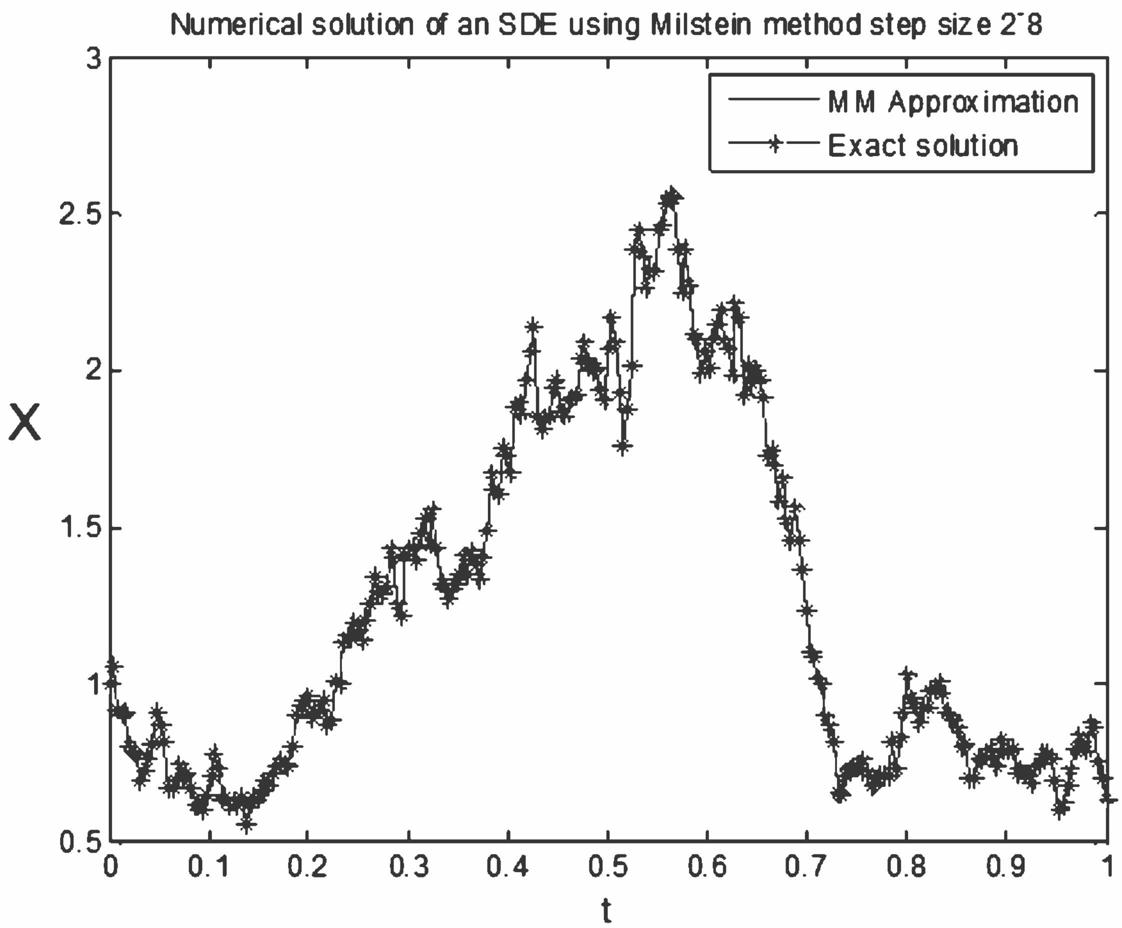


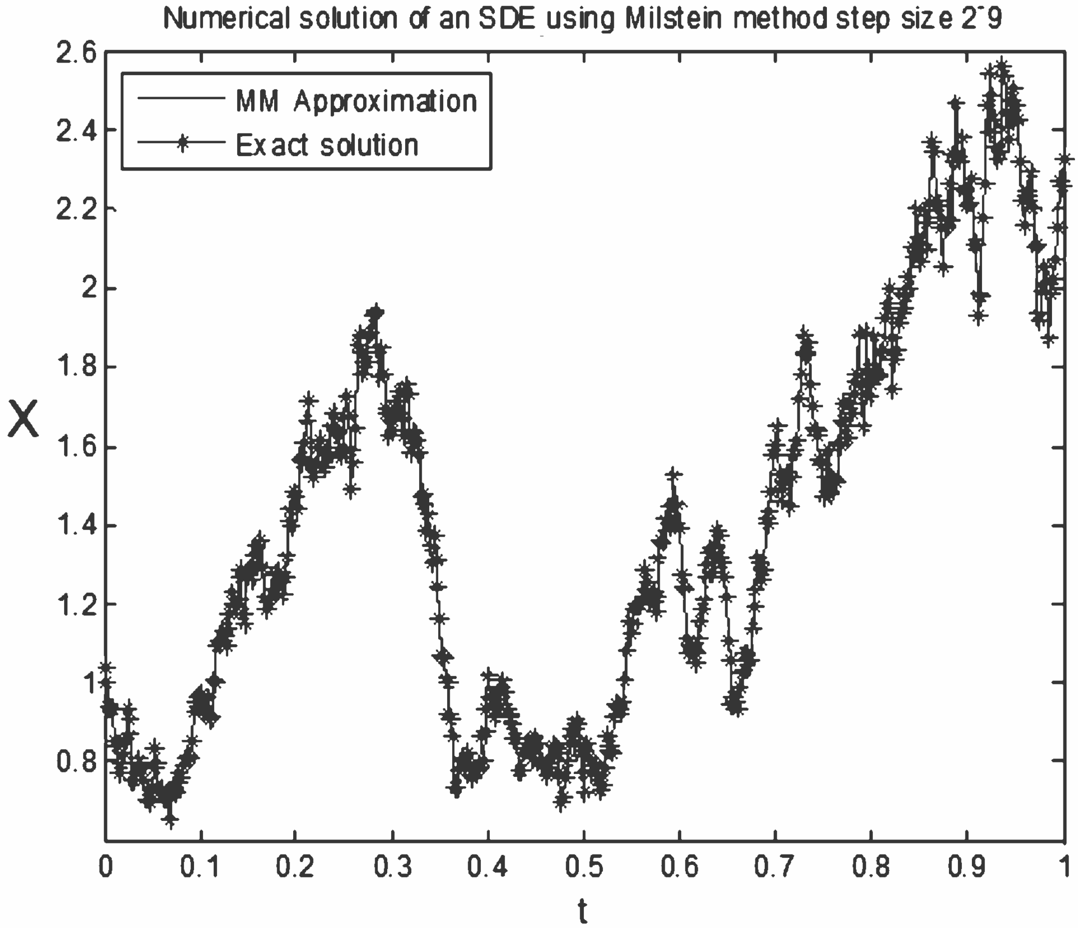
Numerical solution of stochastic differential equations using Milstein method MM 2^{-8}

t-value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.668043545181747	0.669743825973658	0.001700280791911
0.1250	0.626763345346449	0.628596254565663	0.001832909219214
0.1875	0.903685269923684	0.906248304029256	0.002563034105572
0.2500	1.171578411371640	1.174351972463167	0.002773561091527
0.3125	1.480835288132010	1.483730417208125	0.002895129076115
0.3750	1.401052699132930	1.403975769568382	0.002923070435452
0.4375	1.848715225905090	1.852317948621939	0.003602719571039
0.5000	2.065901822124680	2.069717165622714	0.003815343498034
0.5625	2.556748627063660	2.319048075976621	0.237700551087039
0.6250	1.985530788090960	1.407202821142510	0.578327966948450



Numerical solution of stochastic differential equations using Milstein method MM 2^{-9}

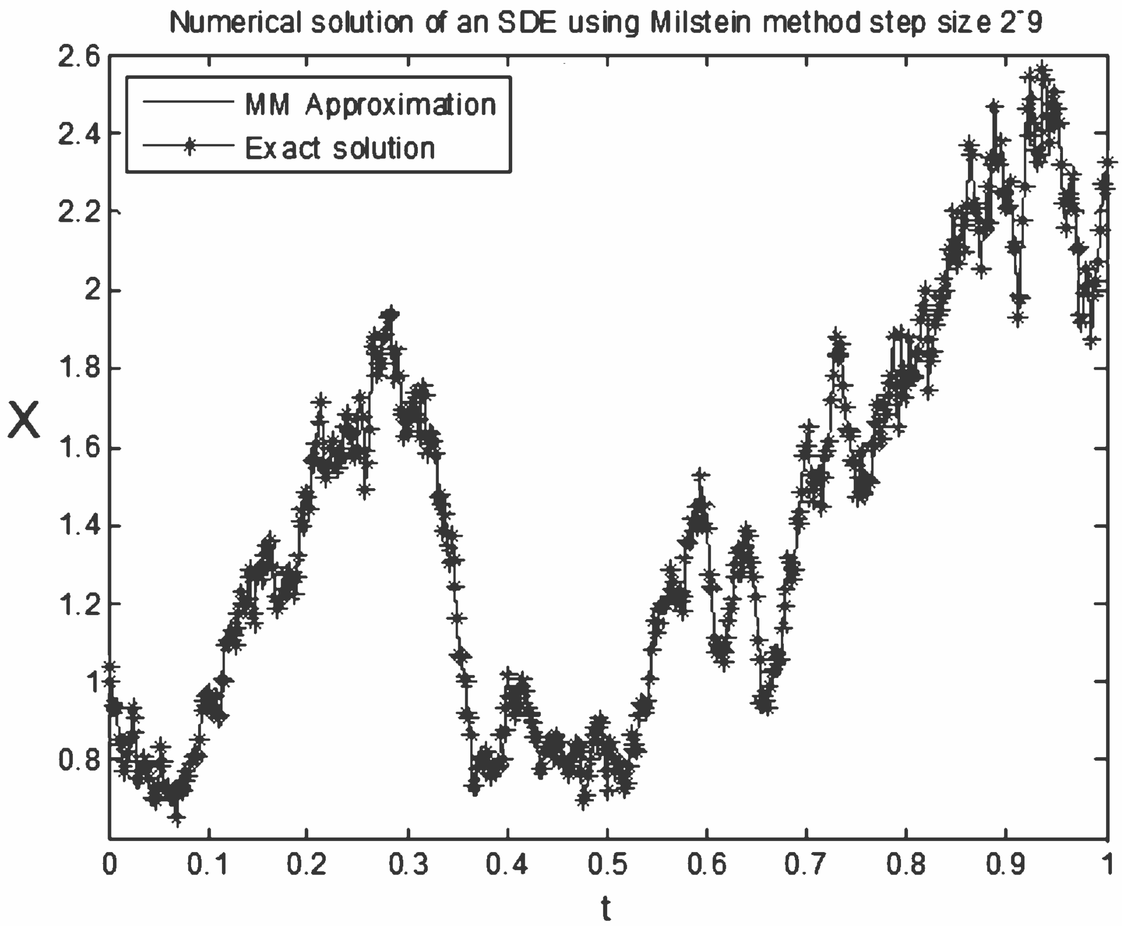
t-value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.718671748167676	0.719378091321411	0.000706343153735
0.1250	1.118481217017121	1.119338967788691	0.000857750771570
0.1875	1.269284421139427	1.270118988866274	0.000834567726847
0.2500	1.670383497022962	1.671319322850989	0.000935825828027
0.3125	1.624166625283062	1.624779761374225	0.000613136091163
0.3750	0.810920442543822	0.812198929625458	0.001278487081636
0.4375	0.823183887302603	0.824401252632072	0.001217365329469
0.5000	0.723620556846645	0.724906216086461	0.001285659239816
0.5625	1.238537625370480	1.239699342708931	0.001161717338451
0.6250	1.208806738431730	1.210349926820979	0.001543188389249



Numerical results for question two using Milstein Method

Numerical solution of stochastic differential equations using Milstein method MM 2^{-5}

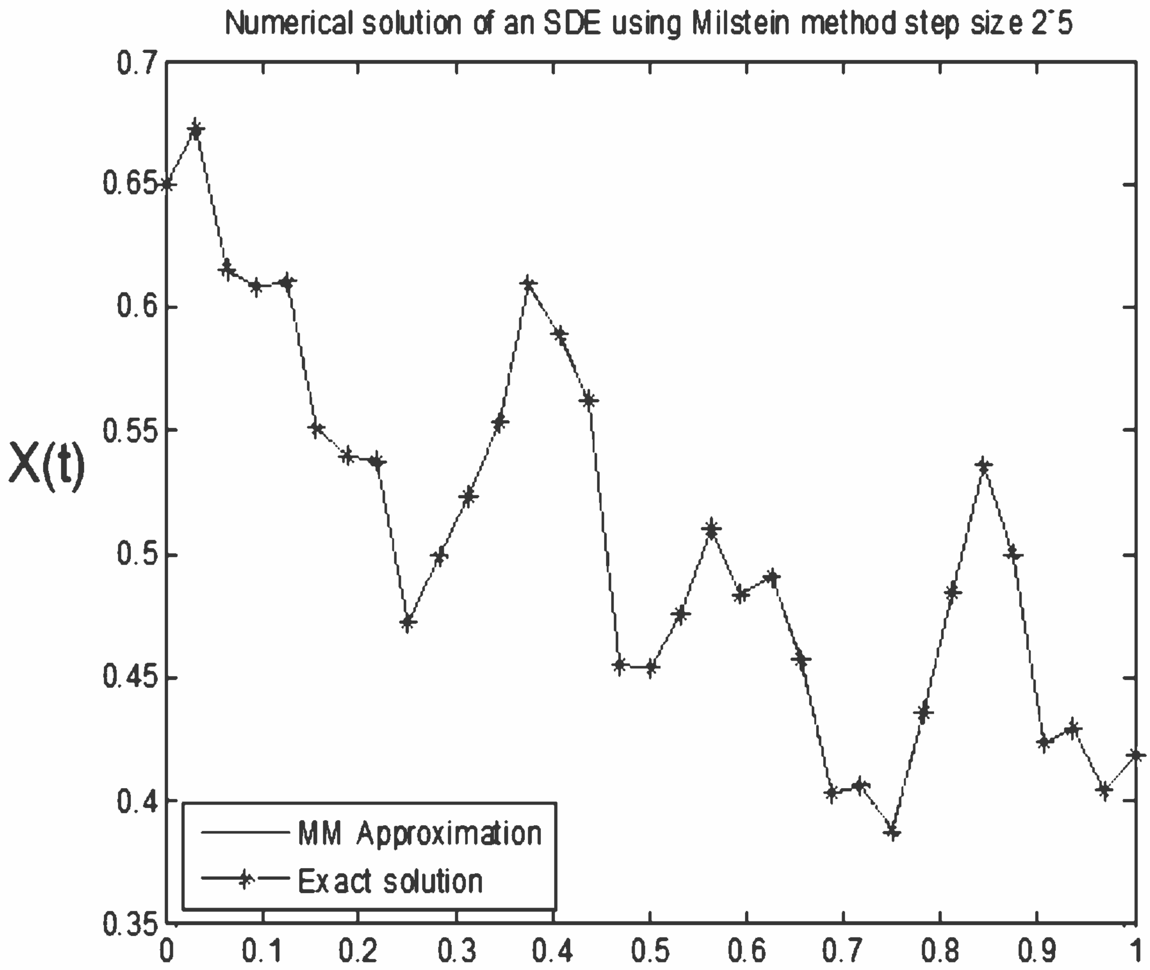
t-value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.615238312843228	0.615287991307386	0.000049678464158
0.1250	0.610538623128736	0.610588793094379	0.000050169965643
0.1875	0.540179430271432	0.540246283393043	0.000066853121611
0.2500	0.472532297247553	0.472575860209679	0.000043562962126
0.3125	0.523183544189210	0.523229546537585	0.000046002348375
0.3750	0.609550392860017	0.609597400120823	0.000047007260806
0.4375	0.561870675494029	0.561922837443626	0.000052161949597
0.5000	0.454256491476679	0.454219214730952	0.000037276745727
0.5625	0.510439943252099	0.510414483967632	0.000025459284467
0.6250	0.490520779248640	0.490492017114674	0.000028762133966



Numerical results for question two using Milstein Method

Numerical solution of stochastic differential equations using Milstein method MM 2^{-5}

t-value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.615238312843228	0.615287991307386	0.000049678464158
0.1250	0.610538623128736	0.610588793094379	0.000050169965643
0.1875	0.540179430271432	0.540246283393043	0.000066853121611
0.2500	0.472532297247553	0.472575860209679	0.000043562962126
0.3125	0.523183544189210	0.523229546537585	0.000046002348375
0.3750	0.609550392860017	0.609597400120823	0.000047007260806
0.4375	0.561870675494029	0.561922837443626	0.000052161949597
0.5000	0.454256491476679	0.454219214730952	0.000037276745727
0.5625	0.510439943252099	0.510414483967632	0.000025459284467
0.6250	0.490520779248640	0.490492017114674	0.000028762133966



Numerical solution of stochastic differential equations using Milstein method MM 2^{-6}

t-value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.622434630313017	0.622451904467868	0.000017274154851
0.1250	0.529935702850741	0.529960393841967	0.000024690991226
0.1875	0.621752369276110	0.621771645094085	0.000019275817975
0.2500	0.517996578796898	0.518016160315794	0.000019581518896
0.3125	0.541740907696983	0.541760913484717	0.000020005787734
0.3750	0.475023745337134	0.475038649472841	0.000014904135707
0.4375	0.547797116183121	0.547822238157476	0.000025121974355
0.5000	0.494277942696807	0.494293794603956	0.000015851907149
0.5625	0.489521447187964	0.489530303495337	0.000008856307373
0.6250	0.529757443757840	0.529766918296393	0.000009474538553

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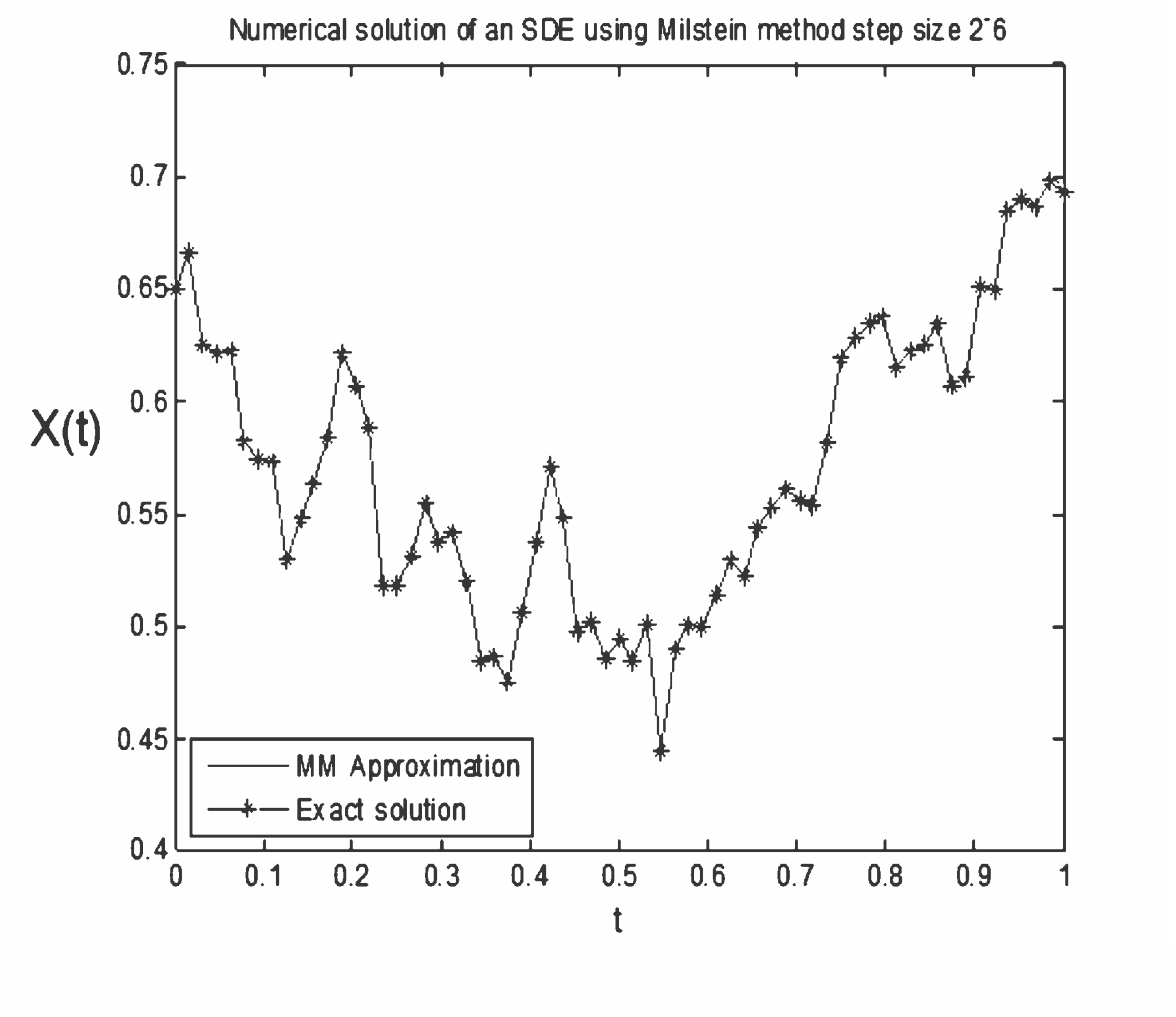
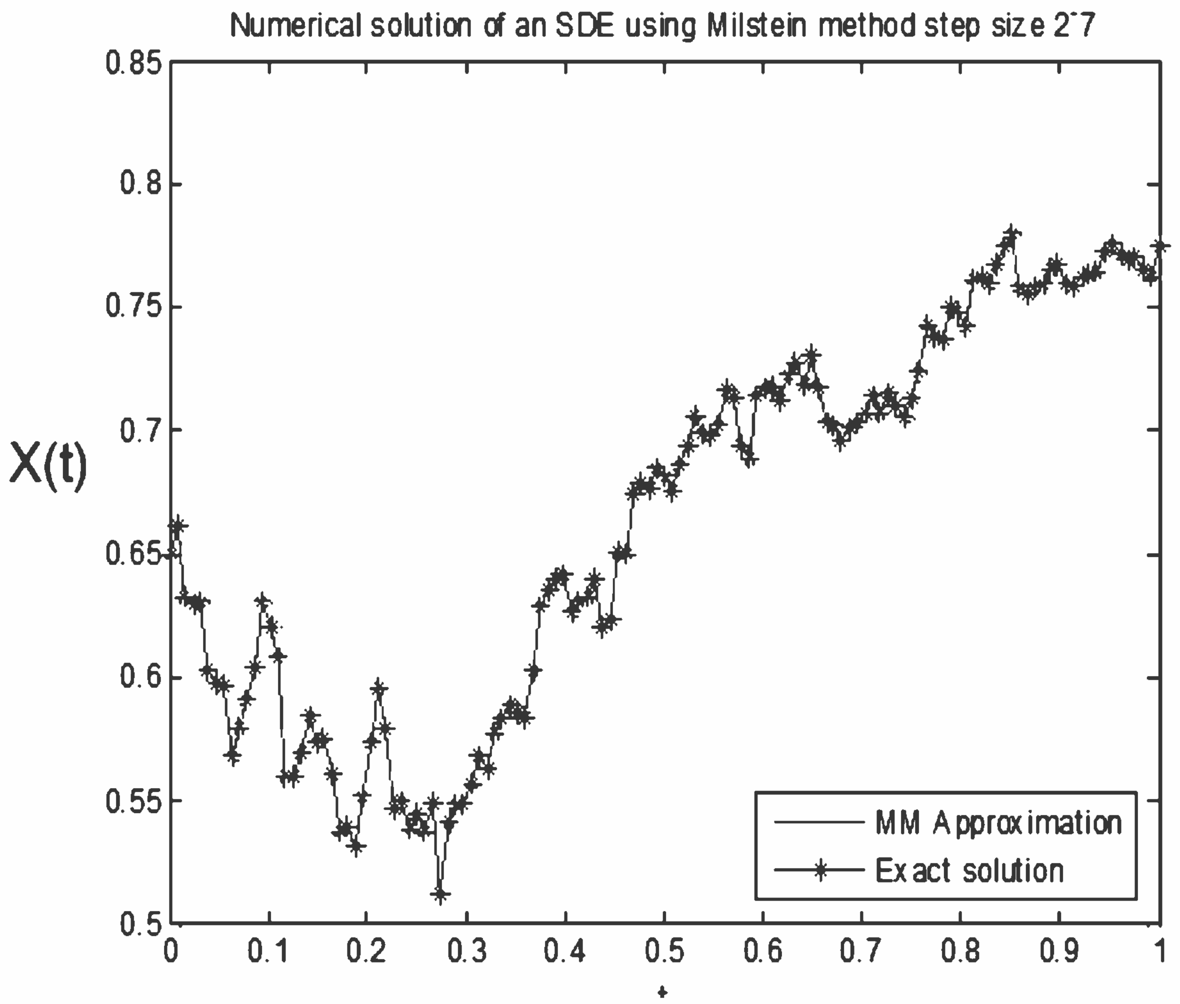


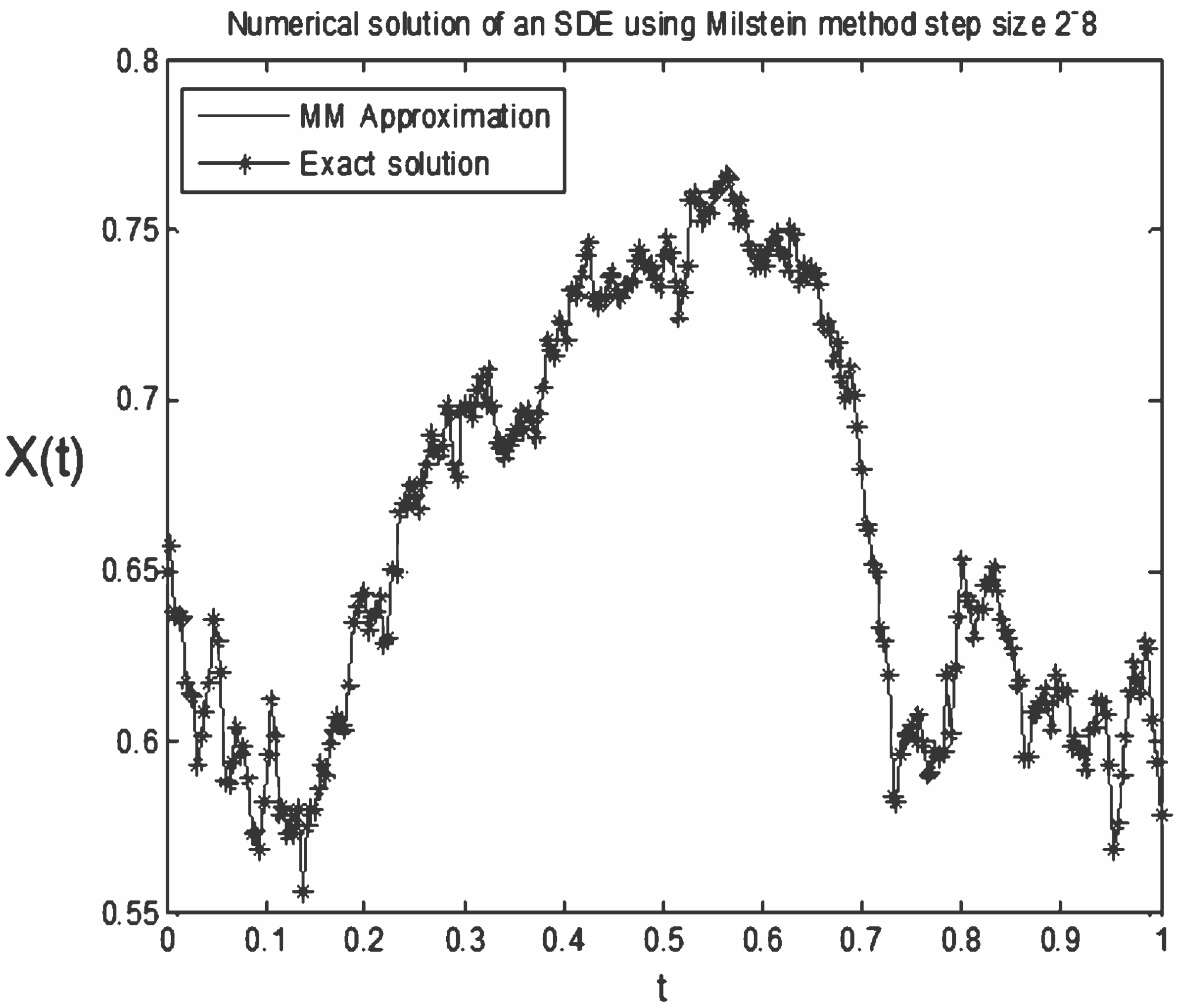
Numerical solution of stochastic differential equations using Milstein method MM 2^{-7}

t-value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.567828497526464	0.567839254255746	0.000010756729282
0.1250	0.559891129276148	0.559905503346062	0.000014374069914
0.1875	0.531567415256637	0.531581791046652	0.000014375790015
0.2500	0.544213974168055	0.544228874990436	0.000014900822381
0.3125	0.567709743837613	0.567723574873731	0.000013831036118
0.3750	0.628838681883073	0.628849095443053	0.000010413559980
0.4375	0.619854508300622	0.619867099861305	0.000012591560683
0.5000	0.681165151248706	0.681167529571030	0.000002378322324
0.5625	0.716774956996478	0.716773301973760	0.000001655022718
0.6250	0.723097174279630	0.723089421237521	0.000007753042109



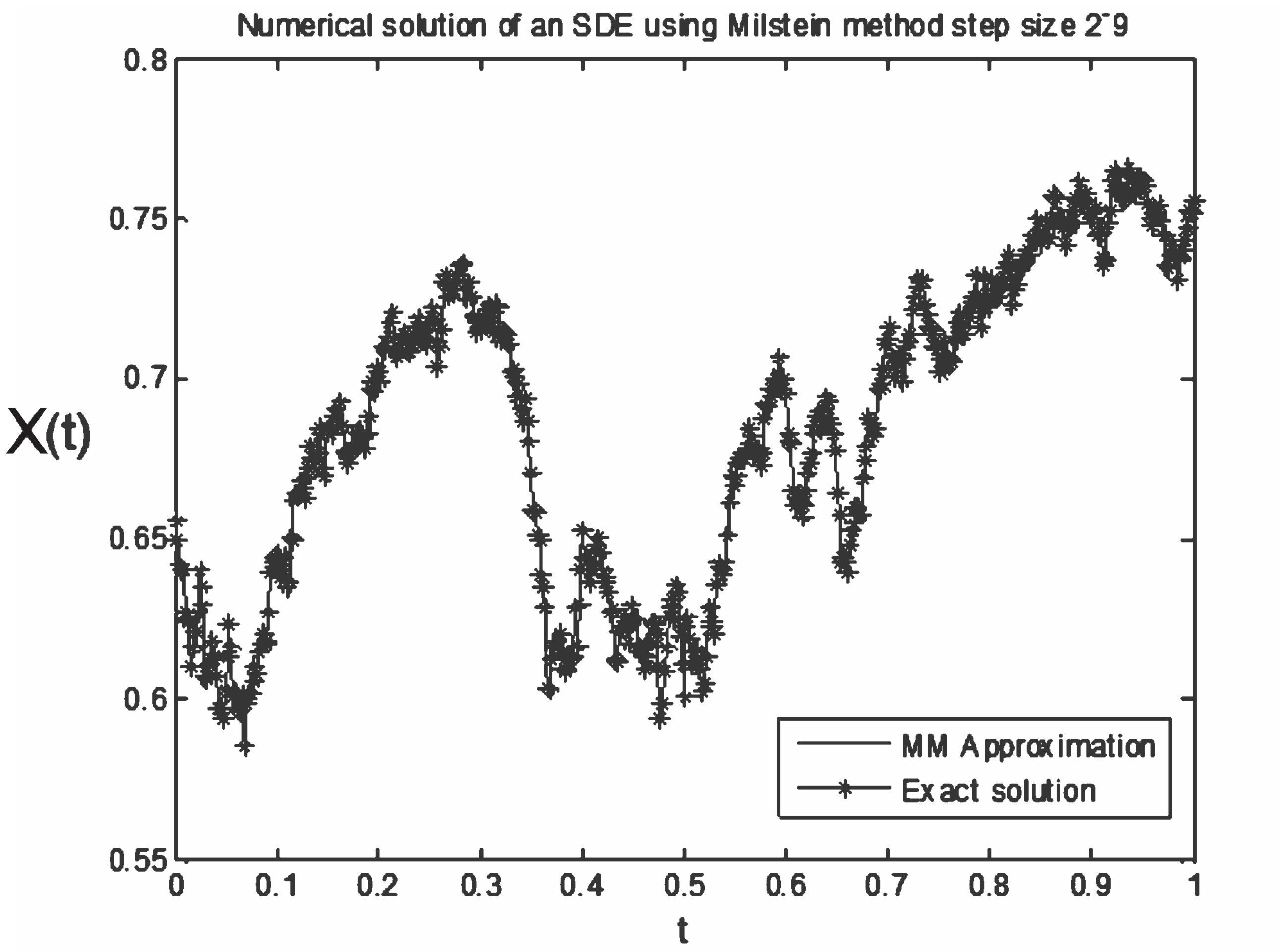
Numerical solution of stochastic differential equations using Milstein method MM 2^{-8}

t-value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.587898306196455	0.587905109023078	0.000006802826623
0.1250	0.577365732605268	0.577373131230515	0.000007398625247
0.1875	0.635137123861844	0.635142812834576	0.000005688972732
0.2500	0.672277179925475	0.672279916436990	0.000002736511515
0.3125	0.703122892986596	0.703122219494964	0.000000673491632
0.3750	0.696053936077163	0.696054073587617	0.000000137510454
0.4375	0.730090638794333	0.730089281287899	0.000001357506434
0.5000	0.742796982140519	0.742794686544875	0.000002295595644
0.5625	0.765754904971199	0.765745779727250	0.000009125243949
0.6250	0.738317390334205	0.738308859764007	0.000008530570198



Numerical solution of stochastic differential equations using Milstein method MM 2^{-9}

t-value	Exact Solution	Numerical Solution	Absolute Error
0.0625	0.599720812990287	0.599723925696370	0.000003112706083
0.1250	0.665872954832284	0.665874441765157	0.000001486932873
0.1875	0.683104724562316	0.683105351008434	0.000000626446118
0.2500	0.718027856294013	0.718027641174807	0.000000215119206
0.3125	0.714612585851372	0.714610452085382	0.000002133765990
0.3750	0.618703881777585	0.618706233598392	0.000002351820807
0.4375	0.621014517645261	0.621016392229899	0.000001874584638
0.5000	0.600818249121088	0.600820867822473	0.000002618701385
0.5625	0.679821210390718	0.679821002550405	0.000000207840313
0.6250	0.676540623493794	0.676542432565840	0.000001809072046



Discussion of result

In this paper we were able to discuss the method of deriving Euler-Maruyama method and Milstein method. This method was applied to two SDEs. The method was used to determine the numerical solution of the given problem. Absolute errors were calculated using the numerical approximation and the corresponding exact solution. Mean absolute error were determined

Conclusion

Euler Maruyama method is the simplest numerical method for solving stochastic differential equation but has slow convergence. The Milstein method is a Taylor method, meaning that it is derived from the truncation of the stochastic Taylor expansion of the solution. In general, Milstein method converges faster to the stochastic solution process and more accurate than Euler Maruyama method as we can see from the above Tables. Hence, Milstein method is hereby recommended for the solution of stochastic differential equations (SDEs).

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