FURTHER RESULTS ON COMBINATORIAL PROPERTIES OF ORDER-PRESERVING ALTERNATING SEMIGROUPS

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ABSTRACT

Let C_n be the charts on $X_n = \{1,2,\Lambda\ ,n\}$, A_n^c be the alternating semigroups, OA_n^c be the subsemigroups of order preserving alternating semigroups and $N(S)$ be the set of nilpotent elements of the semigroups S $= OA_n^c$. Let $G(n;p,m) =| \{a \in S : h(a) = p \wedge f(a) = m\}|$, $G(n;m) =| \{a \in S : f(a) = m\}|$ and $G(n;p) =| \{a \in S : h(a) = p\}|$. In this paper, some combinatorial properties of OA_n^c for two parameters $G(n,p)$ were discuss and their nilpotent elements were also investigated.

Keywords: Semigroups, Charts, Alternating Semigroup, Order-Preserving Alternating Semigroup and Nilpotent Elements.

1.0 INTRODUCTION

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation. A transformation of X is a function from X to itself. Transformation semigroups is one of the most fundamental mathematical objects that occurs in theoretical computer science; where properties of language depend on algebraic properties of various transformation semigroups related to them.

Let $X_n = \{1,2,\Lambda\ ,n\}$, then a (partial) transformation $a : Dom\,a \subseteq X_n \to Im\,a \subseteq X_n$ is said to be full (or total), if Dom $a = X_n$, otherwise it is called strictly partial. The set of all partial transformation on n-object form a semigroup under the usual composition of transformation. It is denoted by P_n when it is partial, T_n when it is full (or total) and C_n when it is partial one-to-one. The elements C_n are usually called Charts. Partial one-to-one transformations are also called subpermutations (Cameron & Deza, 1979). These are the three fundamental transformation semigroups which were introduced by Howie (1995). The semigroup A_n^c which is a subset of C_n is the main object of the study in this paper. The idea of an even permutation has been generalized via path notation, to one-to-one partial transformation(charts) C_n. A transformation a in C_n said to be even if it can be expressed as a product of an even number of transpositional or a product of any number of circuits/paths of odd length.The set of all even charts on X_n, forms **Alternating Semigroups** usually denoted by A_n^c. This class of transformation semigroup was introduced by Lipscomb (1992) and (1996). Many researchers have worked on different classes of transformation semigroups and obtained many results such as Kehinde (2012) studied some algebraic and combinatorial properties of semigroups of injective partial contraction mappings and isometrics of a finite chain, Adeniji studied identity difference transformation semigroups and Adeshola

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(2013) also studied some semigroups of full contraction mapping on a finite chain. Combinatorial and algebraic properties of S_n (symmetric group), A_n (alternating group) and C_n have been studied over a long period and many interesting and delightful results have emerged (see for example Comtet (1974), Cameron (1994), Cameron (2000), Balakrishnan (1995), Laradji & Umar (2007) and Bashir & Umar (2008)). Recently, inspired by the work of Lipscomb (1996), some combinatorial and algebraic properties of alternating semigroup (A_n^c) and and its subsemigroup order preserving alternating semigroup(OA_n^c) have been studied and some interesting and delightful results have emerged (see for example Bakare & Mankanjuola (2013), Bakare et al. (2014) and Bakare & Mankanjuola (2015)). Enumerative problems of an essentially combinatorial nature arise naturally in the study of semigroups of transformations. Many sequences of numbers and triangles of numbers regarded as combinatorial gems like the Stirling numbers used by Howie (1995, pp. 42,96), the factorial used by Munn (1957) and Umar (1993), the binomial used by Howie & Mcfadden, (1990) and Garba (1994), the Fibonacci number used by Howie (1971), Catalan numbers used by Ganyushkin & Mazorchuk (2003) and (2009) and Lah numbers used by Janson & Mazorchak (2005) and Laradji & Umar (2004), etc; have all featured in these enumeration problems. For a nice survey article concerning combinatorial problems in the symmetric inverse semigroup and some of its subsemigroups we refer the reader to Umar (2010). These enumeration problems lead to many numbers in Sloane (http://www.research.att.com/ njas/sequences/.) but there are others that are not yet in it.

This paper modified more on combinatorial properties of order preserving alternating semigroup OA_n^c .

2.0 PRELIMINARY RESULTS

In this section some basic terminology on C_n, A_n^c and OA_n^c will be introduced and some known combinatorial results that shall be needed later are stated.

Let S be the set of OA_n^c on X_n.

Definition 2.1: A *derangement* a is a permutation without fixed points. i.e $a(x) \neq x$.

Definition 2.2: Transposition is a circuit/cycle of length two denoted by $(i\ j)$.

Definition 2.3: Semitransposition is a proper path of length two denoted by $(i\ j]$.

Definition 2.4: A Transpositional is a chart that is either a transposition $(i\ j)$ or a semitransposition $(i\ j]$.

Definition 2.5: A transformation a in C_n said to be even if it can be expressed as a product of an even number of transpositional or a product of any number of circuits/paths of odd length.The set of all even charts on X_n, is forms **Alternating Semigroups** usually denoted by A_n^c.

Definition 2.6: Let $a \in OA_n^c$. Then, the *order-preserving alternating semigroup* on X_n, is

$$OA_n^c = \{a \in OA_n^c : (\forall\, x, y \in Dom a)\, x \le y \Rightarrow xa \le ya\}.$$

Definition 2.7: A transformation a in a semigroup A_n^c is *nilpotent* if there exists a positive integer k such that $a^k = 0$.

Definition 2.8: Let $a \in OA_n^c$. Then the height $h(a) = | Im\, a |$.

Definition 2.9: Let $a \in OA_n^c$. Then, the fix of a is $f(a) = | F(a) |$. where $F(a) = \{x \in Dom\, a : xa = x\}$.

Definition 2.10

$$G(n, p, m) = | \{a \in S : h(a) = p \wedge f(a) = m\} |.$$

Definition 2.11:

$G(n;p) = |\{\alpha \in S : h(\alpha) = |Im\,\alpha| = p\}|$.

Definition 2.12: $G(n;m) = |\{\alpha \in S : f(\alpha) = m\}|$.
In Umar (2010), it is shown that
$|S| = \sum_m G(n;m) = \sum_p G(n;p)$.

Theorem 2.13: Lipscomb (1996), Theorem 25.1
Let $\alpha \in C_n$ have $rank(n-1)$. Then
$\alpha \in A_c^n$ if and only if its completion α^- is an even permutation of N.

Theorem 2.14: Lipscomb (1996),Theorem 25.2 If $\alpha \in C_n$ has rank at most $n-2$,
then $\alpha \in A_c^n$ (All charts of $rank < n-1$ are even).

Theorem 2.15: Lipscomb (1996), Theorem 25.3
Let A_c^n be set of alternating
semigroup on X_n.

Then,

$$|A_c^n| = \frac{n!}{2} + \frac{n^2(n-1)!}{2} + \sum_{p=0}^{n-2} \binom{n}{p}^2 p! \quad \text{for } n \geq 2.$$

Theorem 2.16: Laradji & Umar (2007) Let n be a non-negative integer and
$\alpha \in N(C_n)$ be the element of nilpotent in C_n.
Then,

$$|N(C_n)| = \sum_{p=0}^{n-1} \binom{n}{p}\binom{n-1}{p} p! = |L_{(n,n-r)}|, \textit{for } 0 \leq p \leq n-1.$$

Theorem 2.17: Garba (1994). Let $S = CO_n$. Then

$$G(n;p) = \binom{n}{p}^2.$$

Theorem 2.18: Garba (1994) . Let $S = CO_n$.

Then $|S| = \binom{2n}{n}$ for $n \geq 0$.

Theorem 2.19: Laradji & Umar (2013). Let $S = CO_n$. Then

$G(n;m) = \sum_{j=m}^{n} G(j-1;m-1) b_{n-j}$, where b_n is the number of nilpotent elements in CO_n.

Theorem 2.20: Laradji & Umar (2013). Let $S = CO_n$. Then $G(n;k) = \binom{n+k-1}{k}$

Theorem 2.21: Laradji & Umar (2013) . Let $S = CO_n$. Then

$$G(n;p,k) = \binom{n}{p}\binom{k-1}{p-1}$$

Theorem 2.22: Laradji & Umar (2013). Let $S = CO_n$. Then $|N(CO_n)| = b_n$, which satisfies
$(n+1)b_{n+1} = 2(4n-1)b_n - 3(5n-3)b_{n-1} - 2(2n-1)b_{n-2}; b_0 = 1 = b_1$ and $b_2 = 3$.

2.1 Example of Elements of Alternating Semigroups

The list below shown the elements of A_n^c from n = 1 to 3. These elements were arranged by their height and within each height by their image sets and kernel classes.

THE LIST
For n = 1,
A_1^c has 2 elements.

$|I\eta| = 0$

$K \backslash I\eta$	{0}
∅	( )

$|I\eta| = 1$

$K \backslash I\eta$	{1}
1	$\binom{1}{1}$

For n=2, A_2^c has 4 elements
$|I\eta| = 0$

$K \backslash I\eta$	{0}

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∅	( )

$|I_l| = 1$

$K \backslash I_l$	{1}	{2}
1	$\begin{pmatrix}1\\1\end{pmatrix}$	
2		$\begin{pmatrix}2\\2\end{pmatrix}$

$|I_l| = 2$

$K \backslash I_l$	{1,2,}
1\2	$\begin{pmatrix}1\,2\\1\,2\end{pmatrix}$

For n=3, A_3^c has 22 elements.

$|I_l| = 0$

$K \backslash I_l$	{0}
∅	( )

$|I_l| = 1$

$K \backslash I_l$	{1}	{2}	{3}
1	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\2\end{pmatrix}$	$\begin{pmatrix}1\\3\end{pmatrix}$
2	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}2\\2\end{pmatrix}$	$\begin{pmatrix}2\\3\end{pmatrix}$
3	$\begin{pmatrix}3\\1\end{pmatrix}$	$\begin{pmatrix}3\\2\end{pmatrix}$	$\begin{pmatrix}3\\3\end{pmatrix}$

$|I_l| = 2$

$K \backslash I_l$	{1,2,}	{1,3}	{2,3}
1\2	$\begin{pmatrix}1\,2\\1\,2\end{pmatrix}$	$\begin{pmatrix}1\,2\\1\,3\end{pmatrix}$	$\begin{pmatrix}1\,2\\2\,3\end{pmatrix}$
1/3	$\begin{pmatrix}1\,3\\1\,2\end{pmatrix}$	$\begin{pmatrix}1\,3\\1\,3\end{pmatrix}$	$\begin{pmatrix}1\,3\\2\,3\end{pmatrix}$
2/3	$\begin{pmatrix}2\,3\\1\,2\end{pmatrix}$	$\begin{pmatrix}2\,3\\1\,3\end{pmatrix}$	$\begin{pmatrix}2\,3\\2\,3\end{pmatrix}$

$|I_l| = 3$

$K \backslash I_l$	{1,2,3,}
1/2/3	$\begin{pmatrix}1\,2\,3\\1\,2\,3\end{pmatrix}\begin{pmatrix}1\,2\,3\\2\,3\,1\end{pmatrix}\begin{pmatrix}1\,2\,3\\3\,1\,2\end{pmatrix}$

2.2 **Example of Elements of Order Preserving Alternating Semigroups**

By definition 2.6, the element below were the order preserving alternating semigroups which can be as a subsemigroup of alternating semigroups. They were also arranged by their height and within each height by their image sets and kernel classes.

THE LIST

For n = 1,

OA_1^c has 2 elements.

$|I_l| = 0$

$K \backslash I_l$	{0}
∅	( )

$|I_l| = 1$

$K \backslash I_l$	{1}
1	$\begin{pmatrix}1\\1\end{pmatrix}$

For n=2, OA_2^c has 4 elements

$|I_l| = 0$

$K \backslash I_l$	{0}
∅	( )

$|I_l| = 1$

$K \backslash I_l$	{1}	{2}
1	$\begin{pmatrix}1\\1\end{pmatrix}$	
2		$\begin{pmatrix}2\\2\end{pmatrix}$

$|I_l| = 2$

$K \backslash I_l$	{1,2,}
1\2	$\begin{pmatrix}1\,2\\1\,2\end{pmatrix}$

For n=3, OA_3^c has 16 elements.

$|I_l| = 0$

$K \backslash I_l$	{0}
∅	( )

$|I_l| = 1$

K \ I_1	{1}	{2}	{3}
1	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\2\end{pmatrix}$	$\begin{pmatrix}1\\3\end{pmatrix}$
2	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}2\\2\end{pmatrix}$	$\begin{pmatrix}2\\3\end{pmatrix}$
3	$\begin{pmatrix}3\\1\end{pmatrix}$	$\begin{pmatrix}3\\2\end{pmatrix}$	$\begin{pmatrix}3\\3\end{pmatrix}$

$|I_1 \quad| = 2$

K \ I_1	{1,2,}	{1,3}	{2,3}
1\2	$\begin{pmatrix}1\,2\\1\,2\end{pmatrix}$		$\begin{pmatrix}1\,2\\2\,3\end{pmatrix}$
1/3		$\begin{pmatrix}1\,3\\1\,3\end{pmatrix}$	
2/3	$\begin{pmatrix}2\,3\\1\,2\end{pmatrix}$		$\begin{pmatrix}2\,3\\2\,3\end{pmatrix}$

$|I_1 \quad| = 3$

K \ I_1	{1,2,3,}
1/2/3	$\begin{pmatrix}1\,2\,3\\1\,2\,3\end{pmatrix}$

3.0 MAIN RESULTS

3.1 Combinatorial Results on OA_n^c

Lemma 3.1.1: Let $S = OA_n^c$. Then, $G(n;0) = 1$ for all n.

Proof: It follows from the fact that only the empty mapping will produce zero height.

Lemma 3.1.2: Let $S = OA_n^c$. Then,
$G(n;p_n) = G(n;n) = 1 \forall n.$

Proof: From the listed elements above, there is exactly one order-preserving alternating

semigroup of height n, which is the identity element. $\begin{pmatrix}1 & 2 & - & - & - & n\\ 1 & 2 & - & - & - & n\\ \end{pmatrix}$.

Corollary 3.1.3: Let $S = OA_n^c$. Then,

$$G(n;p_{n-1}) = \frac{1}{4}(-(-1)^n + (1)^n) + \frac{n^2}{2}$$

Proof: The sequences
$G(n;p_{n-1}) = 1, 2, 5, 8, 13, 18$ can be expressed in the following
recurrence relation
$a_n = a_{n-2} + 2(n-1), a_0 = 0, a_1 = 1$ and $n \geq 2.$
Let $a_n = a_n^{(h)} + a_n^{(p)}$, where $a_n^{(h)} = a_{n-2}$ and it solutions are $a_n^{(h)} = a_1(-1)^n + a_2(1)^n$
where a_1 and a_2 are constant. to find $a_n^{(p)}$ since $F(n) = 2(n-1)$ as a trial solution
$a_n = (An+B)n$ where A and B are constant.
By substituting the terms of this sequences into the given recurrence relation,
thus we have,

$$a_n = (An+B)n = A((n-2)+B)(n-2)+2(n-1)$$
$$= (-4A+B+2)n + (4A-2B-2) = 0$$

By solving the two equations we have $B = 0$ and $A = \frac{1}{2}$

therefore, $a_n^{(p)} = (An+B)n = (\frac{1}{2}n+0)n = \frac{n^2}{2}$

$$a_n = a_n^{(h)} + a_n^{(p)}$$
$$= a_1(-1)^n + a_2(1)^n + \frac{n^2}{2}$$

with the initial conditions $a_0 = 0$ and $a_1 = 1$

we have $a_1 = \frac{-1}{4}$ and $a_2 = \frac{1}{4}$

$$a_n = \frac{-1}{4}(-1)^n + \frac{1}{4}(1)^n + \frac{n^2}{2}$$
$$G(n;p_{n-1}) = \frac{1}{4}(-(-1)^n + (1)^n) + \frac{n^2}{2}$$

hence the results.

Theorem 3.1.4: Let $S = OA_n^c$. Then,

$$G(n;p,m_0) = G(n;0) = \begin{cases} b_n, \text{ if n is odd} \\ b_n - 2, \text{ if n is even} \end{cases}$$

Proof:

Let $a=\begin{pmatrix} 2 & 3 & - & - & - & n \\ 1 & 2 & - & - & - & n-1 \end{pmatrix}$ and

$h=\begin{pmatrix} 1 & 2 & - & - & - & n-1 \\ 2 & 3 & - & - & - & n \end{pmatrix}$, then

$g(a)=g(h)=0$. From the definition, there is a unique path of a nilpotent at $h(a)=n-1$ and $h(h)=n-1$ in OA_n^c, then the completion a^- and h^- is odd if n is even and even if n is odd. Hence, the result follows.

Theorem 3.1.5: Let $S=OA_n^c$. Then,

$$G(n;p)=\begin{cases} 1, \text{if } p=n \\ \frac{1}{4}(-(-1)^n+1)+\frac{n^2}{2}, \text{if } p=n-1 \\ \binom{n}{p}^2 \text{ for } 0\le p\le n-2 \end{cases}$$

Proof: If $p=n$ the proof follows from Lemma 3.1.2, if $p=n-1$ it follow from corollary 3.1.3 and for $0\le p\le n-2$ it follow from theorem 2.17 respectively.

Theorem 3.1.6: Let $S=OA_n^c$. Then,

$$|OA_n^c|=\binom{2n}{n}-\binom{n}{n-1}^2+\frac{n^2}{2}-\frac{1}{4}(-1)^n+\frac{1}{4}$$

Proof: It is well known that $\sum_{p=0}^{n}G(n;p)=|OA_n^c|$.

Thus,

$$\sum_{p=0}^{n}G(n;p)=\sum_{p=0}^{n-2}G(n;p)+G(n;p_{n-1})+G(n;p_n)$$

By Theorem 2.17, we have

$$|OA_n^c|=\sum_{p=0}^{n-2}\binom{n}{p}^2+[\frac{1}{4}(-(-1)^n+1)+\frac{n^2}{2}]+1$$

$$=\sum_{p=0}^{n}\binom{n}{p}^2-[\binom{n}{n-1}^2+\binom{n}{n}^2]-\frac{1}{4}(-1)^n+\frac{1}{4}+\frac{n^2}{2}+1$$

$$=\sum_{p=0}^{n}\binom{n}{p}^2-[\binom{n}{n-1}^2+1]-\frac{1}{4}(-1)^n+\frac{1}{4}+\frac{n^2}{2}+1$$

$$=\sum_{p=0}^{n}\binom{n}{p}^2-\binom{n}{n-1}^2-1-\frac{1}{4}(-1)^n+\frac{1}{4}+\frac{n^2}{2}+1$$

$$=\sum_{p=0}^{n}\binom{n}{p}^2-\binom{n}{n-1}^2+\frac{n^2}{2}-\frac{1}{4}(-1)^n+\frac{1}{4}$$

By Theorem 2.18, we have

$$|OA_n^c|=\binom{2n}{n}-\binom{n}{n-1}^2+\frac{n^2}{2}-\frac{1}{4}(-1)^n+\frac{1}{4}$$

Hence, the result follows.

3. 2 Nilpotent Elements in OA_n^c

Theorem 3.2.1: Let $N(OA_n^c)$ be as defined earlier. Then,

$$|N(OA_n^c)|=\begin{cases} b_n, \text{if n is odd} \\ b_n-2, \text{if n is even} \end{cases}$$

where b_n, satisfies the recurrence relation $(n+1)b_{n+1}=2(4n+1)b_n-3(5n-3)b_{n-1}-2(2n-1)b_{n-2}$ with the initial condition $b_0=1=b_1$ and $b_2=3$.

Proof: It can be observed that whenever n is odd all the nilpotent elements in order-preserving partial one-one are also in order-preserving alternating semigroups then it follows by theorem 2.22 For the second case, it follows from theorems 2.13 and 2.14 respectively, showing that whenever n is even the nilpotent elements in A_n^c of rank $n-1$ decrease by 2 elements and the 2 elements are proper paths of length n which are not in A_n^c

hence, the result follows.

Remark 3.2.2: For some computed values of $G(n;p)$ in $|OA_n^c|$ and $N(OA_n^c)$ see Tables 3.2.1 and 3.2.2 below.

Table 3.2.1: $|OA_n^c|$ Elements by their Height

n/p	0	1	2	3	4	5	6	$\sum G(n;p)=\|OA_n^c\|$
0	1							1
1	1	1						2
2	1	2	1					4
3	1	9	5	1				16
4	1	16	36	8	1			62
5	1	25	100	100	13	1		240
6	1	36	225	400	225	18	1	906

Table 3.2.2: $|N(OA_n^c)|$ Elements by their Height

n/p	0	1	2	3	4	5	6	$\sum G(n;p)=\|N(OA_n^c)\|$
0	1							01
1	1							01
2	1	0						01
3	1	6	2					09
4	1	12	14	0				27
5	1	20	50	24	2			97
6	1	30	129	135	36	0	0	331

4.0 CONCLUSION

The combinatorial functions $G(n;p)$ and $G(n;p,m_0)$ were used to derived some triangles of numbers for subsemigroups OA_n^c of alternating semigroups and order of the semigroups were obtained. Further more nilpotent elements of the subsemigroups OA_n^c were also investigated.

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