VORTEX STREETS ON ORIENTABLE SURFACES

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Abstract
We present a dynamics of a set of point vortices on orientable surfaces and investigate the relationship between the streets of single core-rotating row of point vortices and counter-rotating rows of point vortices on a plane surface, as well as on a spherical surface, it was found that itconsist of one or more periodic rows of identical point vortices on each of the surfaces considered. The symmetrical double row and staggered double row was discussed briefly. Finally a conclusion was made by observation, that the same vortex streets on a plane can be transported to the surface of a sphere.

Keywords: Core-rotating Vortices, Counter-rotating Vortices, Orientable Surfaces, Vortex streets.

1.0 Introduction

Theplural of vortex is called vortices, it usuallyform a stirred fluid and may be observed in phenomena such as smoke rings, whirlpool in the wake of boat of the wind surrounding a tornado or dust devil. Vortex often forms around areas of low pressure and attract the fluid and the objects moving within it towards its center.In literature, few exact smooth solutions are known for flow on a sphere, Ghada*et al.* (2011) and Crowdy (2014). The only exact distributed vortex equilibriaas opposed to point verticeson a sphere appear to be the exact solutions on a rotating sphere presented in Crowdy (2014)and two studies by Crowdy, one involving a generalization of Stuart vortices to the sphere.

Definition 1:A point vortex model is a model of flow in which the vorticity is zero everywhere, except at the point itself where the vorticity is infinite, so that there is a nonzero circulation around the pointGhada et al. (2011), Crowdy (2014) and James *et al.* (2014).

Definition 2:A surface $_gS$is said to be orientable, if there exist a unit normal field to every point on the surface. For example Spheres, planes and tori are orientable surfaces but Möbius, real projective planes and Klein bottles are non-orientable.

A design problem for a vortex system in equilibrium with boundaries described by the two-dimensional steady-state Euler equations was solved inBartosze (2012). They introduced a class of steady-state solutions of two-dimensional Euler equations known as the Prandtl–Batchelor flows which is used as a model vortex system, and formulated the vortex design problem mathematically. Also,they introduced elements of the shape calculus and established the optimization framework and discuss some numerical aspects of the solution of the optimization problem. Finally the computational results were presented. The formation of vortex rings generated through impulsively started jetsis studied experimentally in Mortaza*et al.* (1998). Utilizing a piston/cylinder arrangement in a water tank, the velocityand vorticity elds of vortex rings are obtained using Digital Particle Image Velocimetry (DPIV) for a wide range of piston stroke to diameter (*L*=*D*) ratios. The results indicate that the flow eld generated by large *L*=*D* consists of a leading vortex ring followed by a trailing jet. The study of two-dimensional vortex streets and transporting these streets to the sphere was presented in Ghada et al. (2011). Where, two vortex streets were closely examined as a rowof core-rotating vortices and a row of counter-rotating vortices. The Laplacian for cylinder was obtained by using theknown definition of differential operator for cylindrical coordinates$(,,)zrf$in Bitrus *et al* (2018) and Bartosze (2012). It wasdecomposed into three different Ordinary Differential Equations(ODE) and solved by variable separable method. The first two were solved independently and substituted in to the third to get the general solution. Each of the solutions $\cdot(,)Zzz_l$ and $\cdot(,)Q_m ff$ was characterized as a component of the Green's function $\cdot(,)Gxx_\Omega$ which is the stream function for the flow. The parameters z and $\cdot z$ were considered by the angles they form with the $xy plane-$, while l, m and l are constants that assume positive values. Finally the contour plots of the solutions were presented, which describe vortex flow in the interior region of a closed Cylinder of arbitrary length l. Stream function for the Laplace-Beltrami equation on the surface of a three dimensional ring torus were obtained and decomposed in to two different parts namely $()yq_*$ and $()yf_{**}$, and solved independently for the explicit representation of the stream function $(),yqf$ on the torus surface. Their approach is analytic and the result was first of its kind Bitrus *et al* (2018). They also presents some contour plots of the solution obtained by considering different values of q and f as the angles of rotation on $_{,rR}t$ in the interval $[]0,2p$ which illustrates the structure of interesting point vortices on the surface considered. Dynamical system of point vortices on the hyperboloid is investigated in James *et al.* (2014). They provide the classification of relative equilibria and the stability criteria for a number of cases, focusing on two and three vortices. Dissimilarto the system on the sphere, this system has relative equilibria with non-compact momentum isotropy subgroup, and these are used to demonstrate the different stability types of relative equilibria. The study of vortex dynamics on simply connected surfaces of constant curvature K, i.e a plane, spheres and hyperbolic surfaces was done in Dritschel (1988), Polvani and Dritschel (1993). It was known polygonal configuration of N-point vortices are relative Equilibria of the system as a result of these, they study the stability of such configurations and more specifically on how stability depend on the number of point vortices N and the curvature K of the surface. Two-dimensional non-steady viscous flow around a circular cylinder is investigated in Mortaza *et al.* (1998). By solving the exact two-dimensional Navier-Stokes equations numerically for Reynolds numbers (Re) of 40 and 100 and also explained in detail that when Re is 40, a steady state solution can exist. But on the other hand, when Re is 100, flow pattern does not become steady and the Kármán vortex street appears. In the computations performed, these features are successfully apprehended, and the evolution of flow pattern of Kármán vortex is obtained. The study of point vortices on the surface of a sphere was done in Kefas and Ifidon (2016). Crowdy and Cloke (2003). By solving the Laplace-Beltrami equation using a Green's function approach and reconstruct the solutions in spherical polar coordinateswhich leads to obtaining the fundamental Green's function for a sphere.

In this paper our interest is to explore the notion of streets of point vortices on orientable surface (Plane and Sphere) which consist of one or more periodic rows of point vortices and investigate the relationship between single core-rotating vortices and counter-rotating row of point vortices on each of the surfaces and how it was transported from a plane to the surface of a sphere.

2.0 Materials and Methods

Supposed S_g is a general orientable surface, without loss of generality, $S_g \in R^n$ thus any point on can be represented by the conventional notation $P(x,y)$ if $n=2$ and $P(x,h,V)$ if $n=3$ for position of any particle on s_g

Let ω be a vorticity distribution on S_g with the corresponding flow velocity u isincompressible such that $\nabla \cdot u = 0$. Given that ω is conserved, as the fluid particle is induced by ω we can express the velocity field u in term of a suitable differentiable function ψ called streamfunction as;

$$u = n \times \nabla \psi \qquad (1)$$

Where n is a unit normal to the surface under consideration as embedded in R^3. T he scalar vorticity is defined as;

$$\omega = n \cdot \nabla \times u \qquad (2)$$

Upon substituting equation (1) in to equation (2) the stream function seem to satisfy poison equation

$\omega = \nabla^2 \psi$ Wherethe operator ∇^2 is the Laplace-Beltrami operator for the underlined surfaces and we are to solve for the explicit representation of the stream function ψ.

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2.1 The Plane surface

Supposed the surface s_p is a plane surface thenthe governing equation of motion of an inviscid, incompressible fluid in two dimensional hydrodynamic can be written in terms of stream function $\psi(x,y,t)$ as follows

$$\frac{\partial}{\partial t}\nabla^2\psi - \frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)} = 0 \qquad (3)$$

Where $\frac{\partial(pq)}{\partial(x,y)} = \left(\frac{\partial p}{\partial x}\right)\left(\frac{\partial q}{\partial y}\right) - \left(\frac{\partial p}{\partial y}\right)\left(\frac{\partial q}{\partial x}\right)$,is a Jacobian and the operator $\nabla^2 = \left(\frac{\partial^2}{\partial x^2}\right) + \left(\frac{\partial^2}{\partial y^2}\right)$ is the two dimensional Laplacian. The stream function (3) admit the steady state solution of the form

$\nabla^2\psi = \Phi(\psi)$ (4)

For any function Φ, and a number of solution are known for two-dimensional hydrodynamics if we set $\Phi(\psi) = 0$ (4) becomes the two - dimensional Laplace equation $\nabla^2\psi = 0$ with the known f undamental solution of the for

$$\psi = \Gamma \ln d(S, S_o) \qquad (5)$$

Where $d(S, S_o)$ is the distance from the fixed point $x = x_o$ and $y = y_o$ which represent a point vortex of strength Γ at this particular point, for which $\Phi(\psi) = 0$ accept at the point itself where there is nonze ro circulation around the point. Note that the stream function ψ depends only on the geodesic distance i.e $2\pi\psi = \ln(r)$ where $(r \in (0,\infty))$ on a planar surface where the streamfunction has a circular streamlines. The streamfunction is defined in James and Nava-Gaxiola (2014).Mathematically at the j th vortex by limiting process as

$$\psi(s_j,t) = \lim_{s \to s_j}\left(\psi(s,t) - \frac{\Gamma_j}{2\pi}\log d(s,s_j)\right) (6)$$

Where $d(s,s_j) \equiv r$ is the geodesic distance between s and s_j

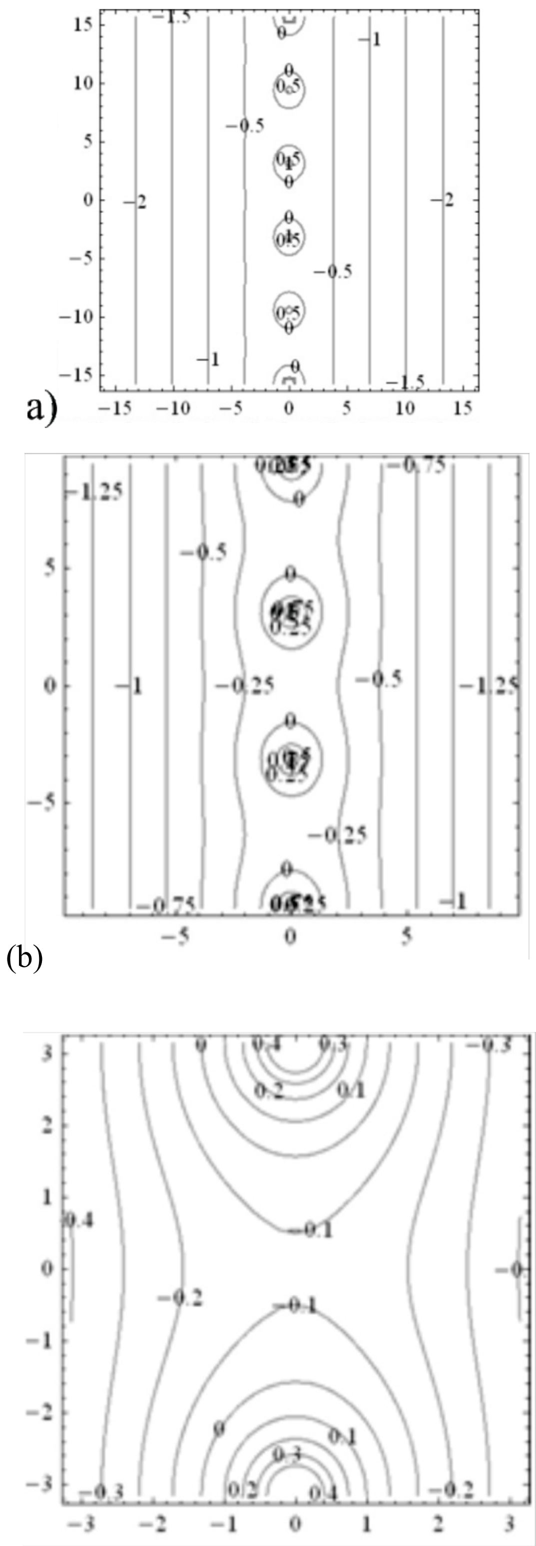

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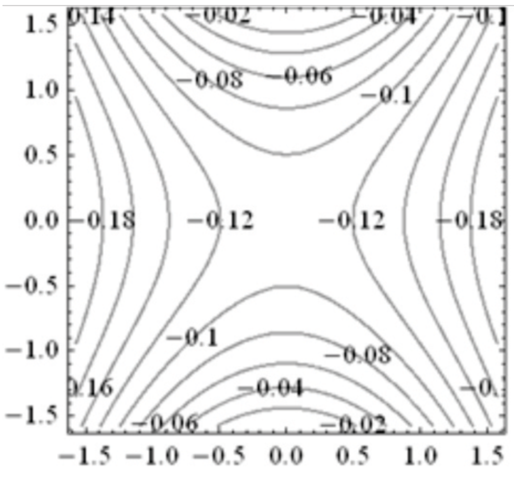
Figure1:*A single core -rotating row of point vortices of strength Γ on a plane with $xy = \pm 5\pi, \pm 3\pi$ and*



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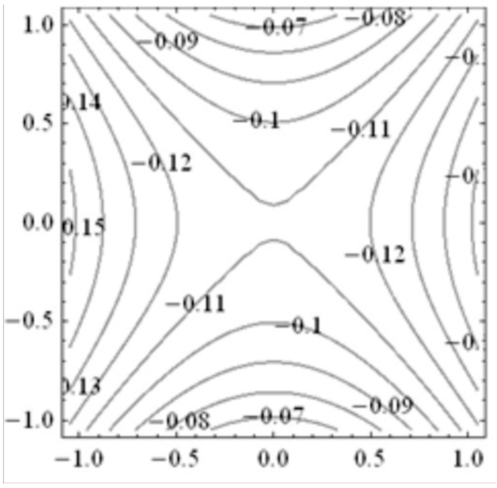

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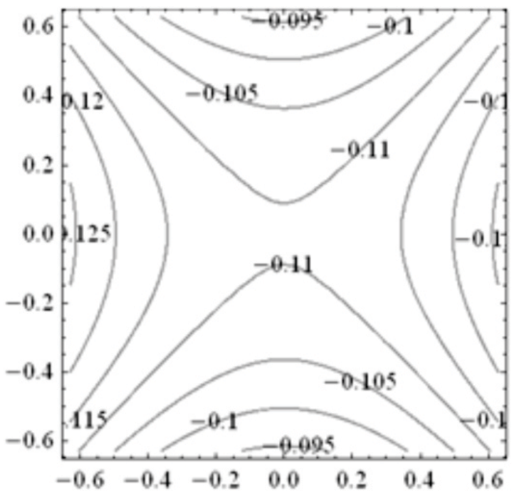

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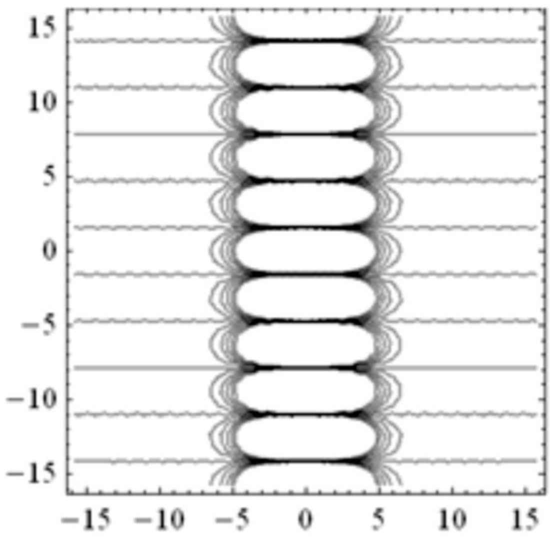
Figure2: *A single core -rotating row of point vortices of strength* Γ *on a plane with* $xy = \pm\pi/5, \pm\pi/3$ *and* $\pm\pi/2$

The presentation in figure1, 2 and 3 are infinite rows of identical point vortices, each of strength Γ analogous to the presentation in [1] positioned at;

$xy = \pm 5\pi, \pm 3\pi$ and $\pm 2\pi$ equally at

$xy = \pm\pi/5, \pm\pi/3$ and $\pm\pi/2$

On combining a row of point vortices of strength Γ and row of point vortices of strength $-\Gamma$ at the same point (xy) we obtained a row of point vortices of opposite sign which is sketch in figure 3 below.



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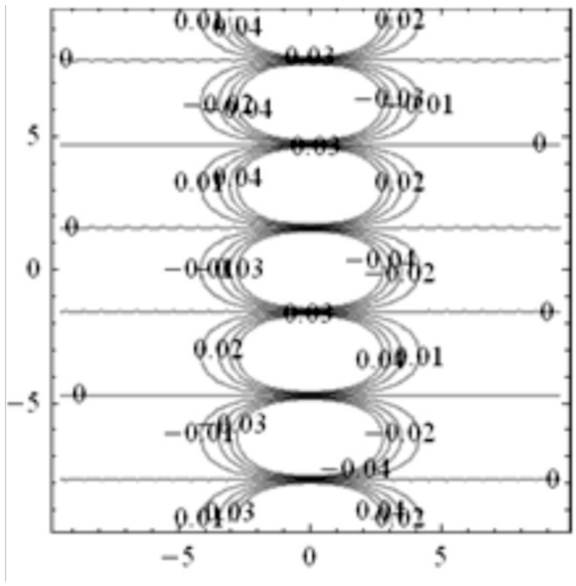

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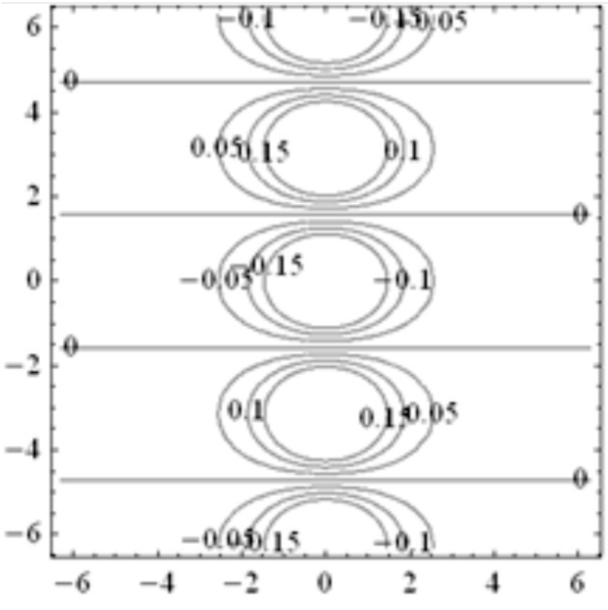

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Figure3: *A ro w of counter rotating point vortices of strength* Γ *on a plane with* $xy = \pm 5\pi, \pm 3\pi$ *and* $\pm 2\pi$

2.1 The Spherical Surface

Supposed the surface under conside ration is a spherical surface, defined by $0 \le \theta \le \pi, 0 \le \phi \le 2\pi$ anddenoted by s_Σ. Here we work in spherical coordinate$(r, \theta\phi)$ and the associated velocity components as $(u_r u_\theta, u_\phi)$ where;

$$u_r = 0 \quad u_\theta = \left(\frac{1}{Sin\theta}\frac{\partial\psi}{\partial\phi}\right) \text{and} u_\phi = -\frac{\partial\psi}{\partial\theta} \quad (7)$$

In terms of the streamfunction$\psi(\theta, \phi)$

The in viscid , incompres sible equation of motion become

$$\frac{\partial}{\partial t}\nabla^2_\Sigma\psi - \frac{1}{R\sin\theta}\frac{\partial(\psi, \nabla^2_\Sigma\psi)}{\partial(\theta, \phi)} = 0 \quad (8)$$

Together with radial pressure gradient

$$\frac{\partial p}{\partial r} = R^{-1}u(u_\theta + u_\phi)$$

$$\nabla^2_\Sigma = \frac{1}{R^2}\left[Cosec\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + Cosec^2\theta\frac{\partial^2}{\partial\phi^2}\right] \quad (9)$$

The equation (9) aboveis known as t he Laplace-Beltrami operator on a sphere Crowdy (2014). The operator admits a steady state solution of the form

$$\nabla^2_\Sigma\psi = \Phi(\psi) \quad (10)$$

$\psi = \psi(xy)$ is a point vortex solution on the plane which is 2π -periodic and equally $\psi = \psi(\theta, \phi)$ will produce a point vortex solution on a sphere since the surface of the sphere can be defined by $0 \le \theta \le \pi, 0 \le \phi \le 2\pi$ whereθ and ϕ are the polar and azimuthal angle respectively

Any point on the sphere can be represented in a spherical coordinate, that is$P(\xi\eta\varsigma)$

$\xi = Sin\theta Cos\phi$, $\eta = Sin\theta Sin\phi$ $\varsigma = Cos\phi$ satisfying $\xi^2 + \eta^2 + \varsigma^2 = 1$ for a unit sphere

In the steady case, the material conservation of vorticity is expressed by [2] as

$$\left(\frac{u_\theta}{\sin\theta}\frac{\partial\psi}{\partial\phi} + u_\phi\frac{\partial}{\partial\theta}\right)w = 0 \quad (11)$$

With the use of (7) above we can write (11) as,

$$\frac{1}{\sin\theta}\left(-\frac{\partial\psi}{\partial\theta}\frac{\partial\omega}{\partial\theta} + \frac{\partial\psi}{\partial\phi}\frac{\partial\omega}{\partial\theta}\right) = 0 \quad (12)$$

2.0 Global constraint on vorticity

Vorticity is defined as the curl of the velocity field. Thus is a measure of local rotation of the fluid. The vorticity which is the curl of the flow velocity is very high in the core region surrounding the axis, and nearly absent in the greater vortex. Pressure within the vortex decreases as the proximity from the axis increases.

Since the surface of a sphere is a closed compact surface, it therefore follows from Gauss' theorem that the integral of the scalar vorticity field over the spherical surface must be zero. Dritschel (1988) Thus we have;

$$\iint_S \omega(s,t)\, d\Omega_s = 0 \quad (13)$$

This is a global constraint on the vorticity distribution. In order to satisfy this constraint and instantaneously have an irrotational flow, each

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point vortex must be counterbalanced by another point vortex on theSphere. By inspection, the single counter-rotating row satisfy this constraint because for these vortex streets the total vorticity on the spherical surface is the sum of the circulations of the point vortices, and for each point vortex, there is a point vortex of opposite strength, so that the circulations sum to zero.

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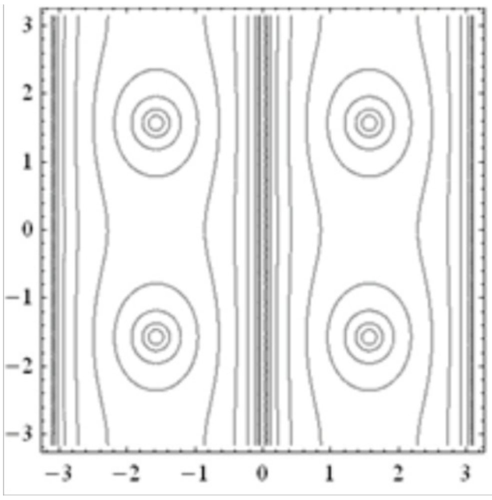

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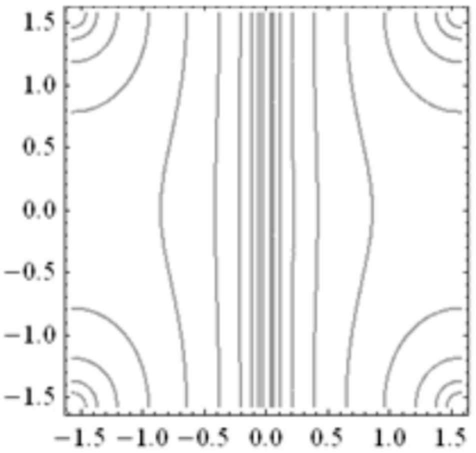

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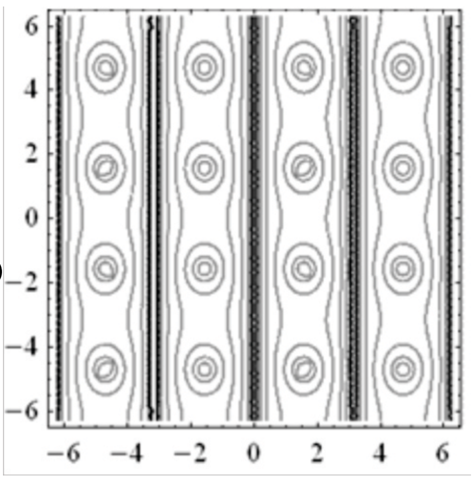
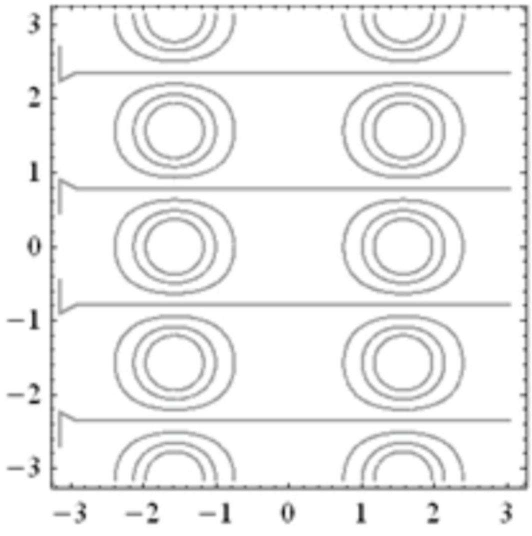


Figure4:*A single row of identical point vortices of strength* Γ *on a sphere with* $\theta, \phi = \pm\pi, \pm\pi/2, \pm 2\pi$ *respectively for* $m \geq 2$

(a)


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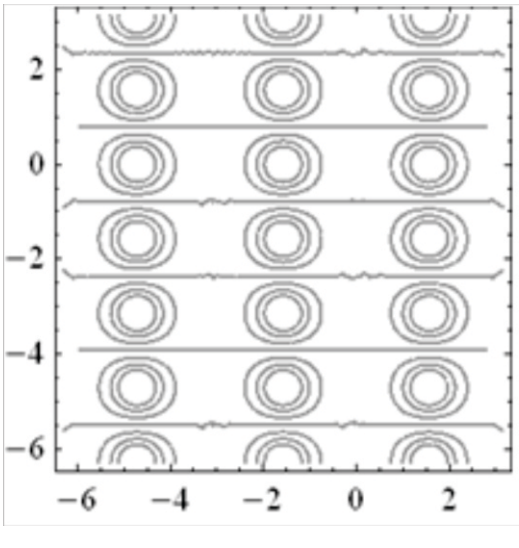

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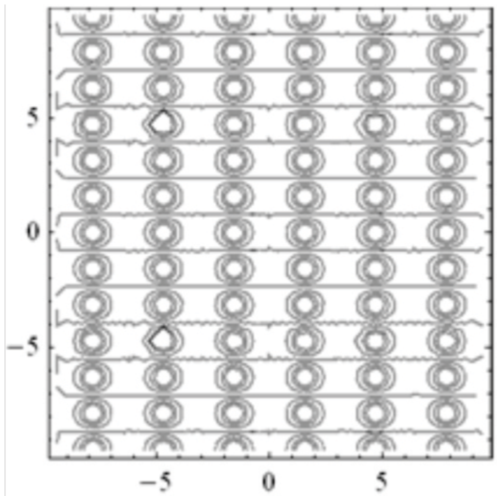

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Figure5: *A ro w of counter rotating point vortices of strength* Γ *on a sphere with* $\theta, \phi = \pm 5\pi, \pm 3\pi$ *and* $\pm 2\pi$ *with* $m \geq 2$

There is also two more combinations d ue to Von Ká rmán as investigated by Ghada and Mallier (2011), which propagate at constant speed in the negative x -direction: the symmetrical double row and the staggered double row both of which consist of two rows of point vortices one along y = d and the other along y = -d with symmetrical double row propagating at speed

$$c = (\Gamma / 2\pi) Coth d \quad (14)$$

And the staggered double row at speed

$$c = (\Gamma / 2\pi) Tanh d \quad (15)$$

4.0 Results and Discussion

Below are the stream functions that described counter-rotating rows of point vortices on a plane and sphere respectively and are potted in figure 1 and 2.

The core-rotating and the counter rotating row of point vortices are found to be stationary, while the symmetrical and the staggeredrows propagate with the given velocities above. The streamfunctions obtained for single core-rotating vortices and single counter-rotating vortices for both plane and spherical surfaces respectively, are listed below.

$$\psi = \frac{\Gamma}{2\pi} \ln\left[Coshy \text{€} osx\right] \quad (16)$$

$$\psi = \frac{\Gamma}{2\pi} \ln\left[\frac{Coshy \text{€} osx}{Coshy \text{€} osx}\right] \quad (17)$$

$$\psi = \frac{\Gamma}{2\pi} \ln\left[Cosh\left[m \ln Tan\left(\frac{\theta}{2}\right)\right] + Cos\left[m\phi\right]\right] \quad (18)$$

$$\psi = \frac{\Gamma}{2\pi} \ln\left[\frac{Cosh\left[m \ln Tan\left(\frac{\theta}{2}\right)\right] + Cos\left[m\phi\right]}{Cosh\left[m \ln Tan\left(\frac{\theta}{2}\right)\right] - Cos\left[m\phi\right]}\right] \quad (19)$$

The stream function for the staggered double row and the symmetrical double row of point vortices on the plane as well as on the sphere respectively are:

$$\psi = \frac{\Gamma}{2\pi} \ln\left[\frac{Cosh(y-d) + Cos(x+ct)}{Cosh(y+d) - Cos(x+ct)}\right] \quad (20)$$

$$\psi = \frac{\Gamma}{2\pi} \ln\left[\frac{Cosh(y-d) + Cos(x+ct)}{Cosh(y+d) + Cos(x+ct)}\right] \quad (21)$$

$$\psi = \frac{\Gamma}{2\pi} \ln\left[\frac{Cosh\left[m \ln Tan\left(\frac{\theta}{2}\right) - d\right] + Cos\left[m(\phi + ct)\right]}{Cosh\left[m \ln Tan\left(\frac{\theta}{2}\right) + d\right] + Cos\left[m(\phi + ct)\right]}\right] \quad (22)$$

$$\psi = \frac{\Gamma}{2\pi} \ln\left[\frac{Cosh\left[m \ln Tan\left(\frac{\theta}{2}\right) - d\right] + Cos\left[m(\phi + ct)\right]}{Cosh\left[m \ln Tan\left(\frac{\theta}{2}\right) + d\right] - Cos\left[m(\phi + ct)\right]}\right] \quad (23)$$

5.0 Conclusion

A point vortex model is a model of flow in which the vorticity is zero everywhere on the surface, except at the point itself where the vorticity is infinite, so that there is a nonzero circulation around the point. As established in this work, flows on orientable surfaces, in particular a closed, compact surface like sphere is very essential as it disclosed so many features applicable to planetary atmospheres, more especially when it comes to vortex flow that has lotsof application in engineering. In general the study of vortex flow is indispensible in human life as we come across it almost everywhere.

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