TWO STEP IMPLICIT BLOCK TAYLOR METHOD FOR THE SOLUTION OF THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT
In this research, an implicit two step block method (ITSBM) for the numerical solution of third order initial value problems of ordinary differential equations has been developed by Taylor expansion technique. The ITSBM developed, when implemented it gave simultaneous solution at every step point. A comparison of our method to an existing method after application to some sample problems reveals that our method was more refined. And also, the ITSBM is self starting, zero stable and consistent.

Keywords: *Block Method, Initial Value Problems, Ordinary Differential Equation, Taylor Expansion, Zero Stable*

1.0 INTRODUCTION

Ordinary differential equations are often used to describe physical systems. The solution of such equations gives valuable insight into how the system evolves and what the effects of changes in the system are. In general, it is extremely difficult, if not impossible, to obtain the analytic solution of such problems. In Science and Engineering, usually mathematical models are developed to help in the understanding of physical phenomena. These models often yield equations that contain some derivatives of an unknown function of one or several variables. Such equations are called differential equations.

Interestingly, differential equations arising from the modelling of physical phenomena often do not have analytic solutions. Hence, the development of numerical methods to obtain approximate solutions becomes necessary. To that extent, several numerical methods such as Finite Difference Methods, Finite Element Methods and Finite Volume Methods, among others, have been developed based on the nature and type of the differential equation to be solved. In most applications of higher orders are solved by reduction to an equivalent system of first order ordinary differential equation, for which an appropriate numerical method to be employed to solve the resultant system. The approach is extensively discussed by some prominent authors (Gragg and Stetter, 1964), (Lambert, 1973)(Butcher, 2003), (Adee, *et al,* 2005), (Awoyemi and Kayode, 2005), (Badmus and Yahaya, 2009) and (Zamurat, *et al,* 2015).

Differential equations are of two types: An Ordinary Differential Equation (ODE) is one for which the unknown function (also known as dependent variable) is a function of a single independent variable. While, a Partial Differential Equation (PDE) is a differential equation in which the unknown function is a function of multiple independent variables and the equation involves its partial derivatives. An ODE is classified according to the order of the highest derivative with respect to the dependent variable appearing in the equation. The most important cases for applications are the first and second order. In particular, Finite Difference Methods have excelled for the numerical treatment of ordinary differential equations especially since the advent of digital computers. The development of algorithms has been largely guided by convergence theorems of (Dahlquist,1963),as well as the treatises of(Fatunla, 1994).

The development of numerical methods for the solution of Initial Value Problems (IVP) of ODE of the form

$$y''' = f\,(x, y, y' y''),\ \ y(\alpha) = yo,\quad y'(\alpha) = \beta,\quad y''(\alpha) = \eta,$$

on the interval $[a,b]$ has given rise to two major discrete variable methods namely; one-step (or single step) methods and multistep methods especially the Linear Multistep Methods (LMM). A numerical method is a difference equation involving a number of consecutive approximations $y'_{n+j}\,,\ j = 0,1,2......,k,$ from which it will be possible to compute sequentially the sequence $\{\,y'_n\ n = 0,1,2......,N\}$

Naturally this difference equation will also involve the function f. The integer k is called the step number of the method. For $k = 1$, it's called a 1-step method and for value of $k>1$ it is it's called a multistep or k-step method.

One step methods include the Euler's methods, the Runge-Kutta methods, the Theta methods, etc. These methods are only suitable for the solutions of first order IVPs of ODEs because of their very low order of accuracy. In order to develop higher order one step methods such as Runge-Kutta methods, the efficiency of Euler's methods, in terms of the number of functional evaluations per step is sacrificed since more valuations is required. Hence, solving (1) using any of the one step methods means reducing it to an equivalent system of first order IVPs of ODEs which increases the dimension of the problem thus increasing its scale. The result is that one step methods become time-consuming for large scale problems and give results that are of low accuracy.

If a computational method for determining the sequence $\{y_n\}$ takes the form of a linear relationship between $y'_{n+j}\,,\ f_{n+j}\,,\ j = 0,1,2...,k,$ we call it a Linear Multistep Method of step number k or a Linear$k$-step method. These methods can be written in the general form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^{\mu} \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (2)$$

Where α_j, β_j are constants and we assume that $\mu = (1,2,3,\ ....)$ an $\alpha_k \neq 0$ Without loss of generality $\alpha_k = 1$ always. Explicit methods are characterized by $\beta_k = 0$ and implicit methods by $\beta_k = 1$ Explicit linear multistep methods are known as Adams-Bashforth methods, while implicit linear multistep methods are called Adams-Moulton methods. These methods are generally called the Adams family. Other famous classes of multistep methods aside the Adams family includes the Predictor-Corrector method and the Backward Differentiation Method.
In this research, we will develop ITSBM for the numerical solution of third order initial value problems of ordinary differential equations on using Taylor expansion technique.

2.0 MATERIALS AND METHODS

We consider the Taylor expansion for

$y_{n+1} = y\,(x_n + h)$ *and* $f_{n+1} = f\,(x_n, = h)$, y'_{n+1} *about* x_n:

$$y_{n+1} = y(x_n + h) y_n + h y'_n + \frac{h^2}{2!} y''_n + \cdots + \frac{h^q}{q!} y_n^{(q)} + \cdots \qquad (3)$$

Where

$$y_n^{(q)} = \frac{d^q y}{dx^q}|_{x=x_n},\ q = 1,2,\ldots \qquad (4)$$

$$f_{n+1} = f(x_n + h) =$$

$$= y'_n + h y''_n + \frac{h^2}{2!} y'''_n + \cdots + \frac{h^q}{q!} y_n^{(q+1)} + \cdots \qquad (5)$$

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Constructing an implicit linear two step method of order three, containing one free parameter of the form (2) where $k = 2$ and always $\alpha_k = \alpha_2 = 1$.
Let $\alpha_0 = a$, be the free parameter.
We now have the general form of the method expressed as;

$$y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n = h^3[\beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n] \quad (6)$$

Since $\alpha_2 = 1$ and $\alpha_0 = \alpha$ are already known. Equation (6) implies,

$$y_{n+2} + \alpha_1 y_{n+1} + \alpha y_n - h^3[\beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n] = 0 \quad (7)$$

Then, the remaining undetermined parameters are $\alpha_1, \beta_0, \beta_1$ and β_2
Now on using Taylor expansion for (7), we have,

$$y_{n+2} = \left[y_n + 2hy_n' + 2h^2 y_n'' + \frac{4}{3}h^3 y_n''' + \frac{2}{3}h^4 y_n^{iv} + \frac{4}{15}h^5 y_n^{v} + \frac{4}{45}h^6 y_n^{vi} + \ldots\right] \quad (8)$$

$$\alpha_1 y_{n+1} = \alpha_1\left[y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3}{6}y_n''' + \frac{h^4}{24}y_n^{iv} + \frac{h^5}{120}y_n^{v} + \frac{h^6}{720}y_n^{vi} + \cdots\right] \quad (9)$$

$$ay_n = ay_n \quad (10)$$

$$h^3\beta_2 f_{n+2} = \beta_2\left[h^3 y'_n + 2h^4 y_n'' + 2h^5 y_n''' + \frac{4}{3}h^6 y_n^{iv} + \frac{2}{3}h^7 y_n^{v} + \frac{4}{15}h^8 y_n^{vi} + \cdots\right] \quad (11)$$

$$h^3\beta_1 f_{n+1} = \beta_1\left[h^3 y'_n + h^4 y_n'' + \frac{h^5}{2}y_n''' + \frac{h^6}{6}y_n^{iv} + \frac{h^7}{24}y_n^{v} + \frac{h^8}{120}y_n^{vi} + \cdots\right] \quad (12)$$

$$h^3\beta_0 f_n = h^3\beta_0 y_n' \quad (13)$$

On comparing (8) to (13), we obtain the following

$$1 + \alpha_1 + a = 0 = C_0$$
$$\alpha_1 = -1 - a \quad (14)$$
$$2 + \alpha_1 - \beta_2 - \beta_1 - \beta_0 = 0 = C_1$$
$$\alpha_1 - \beta_0 - \beta_1 - \beta_2 = -2 \quad (15)$$
$$2 + \frac{1}{2}\alpha_1 - 2\beta_2 - \beta_1 = 0 = C_2$$
$$\alpha_1 - 2\beta_1 - 4\beta_2 = -4 \quad (16)$$
$$\frac{4}{3} + \frac{1}{6}\alpha_1 - 2\beta_2 - \frac{1}{2}\beta_1 = 0 = C_3$$
$$\alpha_1 - 3\beta_1 - 12\beta_2 = -8 \quad (17)$$
$$\frac{2}{3} + \frac{1}{24}\alpha_1 - \frac{4}{3}\beta_2 - \frac{1}{6}\beta_1 = 0 = C_4$$
$$\frac{1}{24}(\alpha_1 - 4\beta_1 - 32\beta_2 + 16) = C_4 \quad (18)$$

From (14) to (17) we obtain a four system of equation given as follows:

$$\alpha_1 = -1 - a \quad (19)$$
$$\alpha_1 - \beta_0 - \beta_1 - \beta_2 = -2 \quad (20)$$
$$\alpha_1 - 2\beta_1 - 4\beta_2 = -4 \quad (21)$$
$$\alpha_1 - 3\beta_1 - 12\beta_2 = -8 \quad (22)$$

Using the maple software package to solve the system (19) to (22) we obtain the following:

$$\left\{\begin{array}{ccc} \alpha_0 = a, & \alpha_1 = -1 - a, & \alpha_2 = 1, \\ \beta_0 = -\frac{1}{12}(1 + 5a), & & \beta_1 = \frac{8}{12}(1 - a), \\ & \beta_2 = \frac{1}{12}(5 + a) & \end{array}\right\}$$

On substituting the values into (8) we obtain

$$y_{n+2} - (1 + a)y_{n+1} + ay_n = \frac{h^3}{12}[(5 + a)f_{n+2} + 8(1 - a) f_{n+1} - (1 + 5a)f_n] \quad (23)$$

Now on generating the methods by putting a = -56 and 61 into (23), we obtain the following two schemes;

at $a = -56$

$$y_{n+2} + 55y_{n+1} - 56y_n = \frac{h^3}{12}[-51f_{n+2} + 456f_{n+1} + 279f_n] \quad (24)$$

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at $a = 61$

$$y_{n+2} - 62y_{n+1} + 61y_n = \frac{h^3}{12}[66f_{n+2} - 480f_{n+1} - 306f_n] \qquad (25)$$

On using (24) and (25) as a block we have

$$y_{n+2} + 55y_{n+1} - 56y_n = \frac{h^3}{12}[-51f_{n+2} + 456f_{n+1} + 279f_n]$$

$$y_{n+2} - 62y_{n+1} + 61y_n = \frac{h^3}{12}[66f_{n+2} - 480f_{n+1} - 306f_n] \qquad (26)$$

3.0 RESULT AND DISCUSSION

The block method has the following order and error constants. The method as a block is of order $\begin{bmatrix} \frac{24}{55} & -\frac{12}{31} \end{bmatrix}^T$ as shown in Table 1

Table 1: Order and Error Constant of the Method.

Method	Order	Error Constant
$y_{n+2} + 55y_{n+1} - 56y_n = \frac{h^3}{12}[-51f_{n+2} + 456f_{n+1} + 279f_n]$	3	$\frac{24}{55}$
$y_{n+2} - 62y_{n+1} + 61y_n = \frac{h^3}{12}[66f_{n+2} - 480f_{n+1} - 306f_n]$	3	$-\frac{12}{31}$

3.1 Convergence Analysis of ITSBM

The convergence analysis of ITSBM was determined using the approach in (Fatunla, 1985 and 1994)

According to Fatunla, a block method can be defined as follows:

Let Y_m and F_m be vectors defined by

$$Y_m = \begin{bmatrix} y_n \\ \vdots \\ y_{n+r-1} \end{bmatrix} \qquad (27)$$

$$F_m = \begin{bmatrix} f_n \\ \vdots \\ f_{n+r-1} \end{bmatrix} \qquad (28)$$

Then a general k-block, r-point method is a finite difference equation of the form

$$Y_m = \sum_{i=1}^{k} A_i Y_{m-i} + h^3 \sum_{i=0}^{k} B_i F_{m-i} \qquad (29)$$

Where all the Ai and Bi are properly chosen $r \ X \ r$ matrix coefficients and $m = 0,1,2,\ldots$ represents the block number, $n = mr$ the first step number in the m-th block and r is the proposed block size.

Where according to (Chu and Hamilton, 1987), the zero stability condition of the block is given as follows:

The block method is said to be zero-stable if the roots $R_j, j = 1\ (1)\ k$ of the first characteristics polynomial $p\ (R)$=det $\left[\sum_{i=0}^{k} A_i R^{k-i}\right] = 0, A_0 = -I,$ satisfies $|R_j| \leq$ 1. If one of the roots is $+1$, we call this root the principal root of $p(R)$.

The method now is expressed in the form of (29) as

$$\begin{bmatrix} 55 & 1 \\ -62 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 56 \\ 0 & -61 \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} + h^3 \begin{bmatrix} \frac{456}{12} & -\frac{51}{12} \\ -40 & \frac{11}{2} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix} + h^3 \begin{bmatrix} 0 & \frac{277}{12} \\ 0 & -\frac{51}{2} \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} \qquad (30)$$

For

$$A^0 = A^{-1}.A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (31)$$

$$A^1 = A^{-1}.B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad (32)$$

$$A^{-1} = \begin{bmatrix} \frac{1}{177} & -\frac{1}{177} \\ \frac{62}{177} & \frac{55}{177} \end{bmatrix} \quad (33)$$

The first characteristics polynomial of the block method is given by

$$\rho(\lambda) = \det(\lambda A^0 - A^1) \quad (34)$$

Substituting the A^0 and A^1 into the equation above gives

$$\rho(\lambda) = det\left[\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right] \quad (35)$$

$$= det\begin{bmatrix} \lambda & -1 \\ 0 & \lambda - 1 \end{bmatrix}$$

$$\Rightarrow \lambda(\lambda - 1) = 0 \quad (36)$$

This will yield

$\lambda = 0, \quad \lambda = 1$

Which show that the method is zero-stable.
By definition (29), the method is zero-stable and consistent since the order of the method $p = 3 > 1$, which implies that the method is Convergent.

3.2 Application of the Method

The constructed ITSBM is tested on some initial value problems of order three differential equations. The results obtained from using these methods are compared with the exact solution.
The proposed methods were applied to the following third order differential equations.

Problem 1

Consider the initial value problem $y''' + 5y'' + 7y' + 3y = 0, y(0) = 1, y'(0) = 0, y''(0) = -1, h = 0.1$, whose exact solution is $y(x) = e^{-x} + xe^{-x}$

The results obtained are tabulated in Table 2. This problem can also be found in (Sagir, 2014).

Problem 2

Consider the initial value problem y''' - y' + $y' - y = 0, y(0) = 1, y'(0) = 0, y''(0) = -1, h = 0.01$, whose exact solution is $y(x) = \cos x$. The results obtained are tabulated in Table 3. This problem can also be found in (Sagir, 2014).

Table 2: Comparison of ITSBM with Sagir for Problem 1.

x	Exact Value	Approximate Value of Sagir	Approximate Value of ITSBM	Error of Sagir	Error of ITSBM
0.1	0.9953211598	0.9953212241	0.995321158	6.4300??- 08	1.0800??- 09
0.2	0.9824769037	0.9824768765	0.982476902	2.7200??- 08	1.7000??- 09
0.3	0.9630636869	0.9630636564	0.963063667	3.0500??- 08	1.9900??- 08
0.4	0.9384480644	0.9384481542	0.938448042	8.9800??- 08	2.2400??- 08
0.5	0.9097959895	0.9097955469	0.909795903	4.4260??- 07	8.6500??- 08
0.6	0.8780986178	0.8780978452	0.878098607	7.7260??- 07	1.0800??- 08
0.7	0.8441950165	0.8441930642	0.844194642	1.9523??- 06	3.7450??- 07
0.8	0.8087921354	0.8087911080	0.808791281	1.0274??- 06	8.5440??- 07
0.9	0.7724823534	0.7724810025	0.772480900	1.3509??- 06	1.4534??- 06
1.0	0.7357588824	0.7357454120	0.735756262	1.3470??- 05	2.6204??- 06

Table 3: Comparison of ITSBM with Sagir for Problem 2.

x	Exact Value	Approximate Value of Sagir	Approximate Value of ITSBM	Error of Sagir	Error of ITSBM
0.01	0.999999984	0.9999502003	0.9999501982	1.9990??- 07	1.878832450??- 09
0.02	0.999999939	0.9998002023	0.999999901	1.9560??- 07	3.80765177??- 08
0.03	0.999999862	0.9995501702	0.999999915	1.3651??- 07	5.20778355??- 08
0.04	0.999999756	0.9992003588	0.999999710	2.5210??- 07	4.56939255??- 08
0.05	0.999999619	0.9987515643	0.999999234	1.3039??- 06	3.85228255??- 07
0.06	0.999999451	0.9982035679	0.999999237	3.0280??- 06	2.140188695??- 07
0.07	0.999999253	0.9975543456	0.999998261	3.3453??- 06	9.92687415??- 07
0.08	0.999999025	0.9968004658	0.999998242	1.2405??- 06	7.83224416??- 07
0.09	0.999998766	0.9959540620	0.999995337	1.3290??- 06	3.42929974??- 06
0.10	0.999998476	0.9950213456	0.999990278	1.7180??- 05	8.198913288??- 06

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3.3 Test of Convergence of ITSBM using Problems 1 and 2

On using Contraction Mapping Theorem,

$$Lh|\beta_k| < 1 \tag{37}$$

in order to ensure convergence of the iteration; here ??is the Lipschitz constant of the function $f(x, y, y', y'')$. In fact, since the function $f(x, y, y', y'')$ is continuously differentiable,

$$\Rightarrow L = \max_{(x,y)\in R} \left\| \frac{\partial f}{\partial x}(x, y) \right\| \tag{38}$$

From (23) which is the general form,

$$\beta_k = \beta_2 = \frac{h^3}{12}(5 + a) \tag{39}$$

Also from problem 1,

$$y''' + 5y'' + 7y' + 3y = 0 \tag{40}$$

$$\Rightarrow f = y''' = -5y'' - 7y' - 3y$$

$$\frac{\partial f}{\partial x} = 0 = L \tag{41}$$

Since $h = 0.1$

$$\Rightarrow Lh|\beta_k| = 0 \times \frac{h^4}{12}(5 + a) \tag{42}$$

$$= 0 < 1$$

This shows that the method is convergent $\forall$ values of a.

On using (37) and (38),
From (23) which is the general form,

$$\beta_k = \beta_2 = \frac{h^3}{12}(5 + a) \tag{43}$$

Also from problem 2,

$$y''' - y'' + y' - y = 0$$

$$\Rightarrow f = y''' = y'' - y' + y \tag{44}$$

Since $h = 0.1$

$$\Rightarrow Lh|\beta_k| = 0 \times \frac{h^4}{12}(5 + a)$$

$$= 0 < 1$$

This shows that the method is convergent $\forall$ values of a.

4.0 CONCLUSION

In this work, we presented ITSBM that are zero stable, consistent and convergent. We used the method to solve numerically IVPs. The results of the presented test problems shows that our method is capable for solving IVPs which is reliable as it generates results that competes favorably with their exact solutions and solutions from Sagir as shown in Table 1 and Table 2.

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