Vol. 1, No. 2, Nov. 2018
ISSN: 2616-1303 | **Web:***www.cjpas.fulokoja.edu.ng*

SUBORDINATION RESULTS ON CERTAIN NEW SUBCLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT
The object of this study is to obtain certain subordination results by making use of a generalized differential operator and some results obtained for certain classes of analytic functions defined by the generalized differential operator. In addition, we also employ Subordinating Factor Sequence and subordination principle. Furthermore, we pose several results on subordination theorem and derive some consequences of our results. Some earlier known results turn out to be special cases of our new results.

Keywords:subordination, analytic function, convex, subordinating factor, convolution.

1.0 INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k . \tag{1}$$

which are analytic in the open unit disk $U = \{z : z \in C : |z| < 1\}$ in the complex plane C and normalized by $f(0) = 0$ and $f'(0) = 1$. Let S and $K(\alpha)$ $(0 \leq \alpha < 1)$ denote subclasses of A consisting of functions that are univalent and convex of order α in U respectively. In particular, the class $K(0) = K$ is the familiar class of convex functions in U. The class of convex functions is defined by

$$K = \left\{ f \in A : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in U \right\}. \tag{2}$$

Definition 1 (Hadamard Product or Convolution). Given two functions $f(z)$ and $g(z)$ where $f(z)$ is as defined in (1) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k , \tag{3}$$

the Hadamard Product (or Convolution) $g * f$ of $g(z)$ and $f(z)$ is defined by

$$(g * f)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (f * g)(z) \tag{4}$$

Definition 2 (Subordination Principle).
Let $f(z)$ and $g(z)$ be analytic in the unit disk U. then $g(z)$ is said to be subordinate to $f(z)$ in U denoted by

$$g(z) \prec f(z), z \in U,$$

if there exists a Schwarz function $w(z)$, analytic in U with $w(0)$, $|w(z)| < 1$ such that

$$g(z) = f(w(z)), z \in U \tag{5}$$

In particular, if the function $f(z)$ is univalent in U, then $g(z)$ is subordinate to $f(z)$ if

$$g(0) = f(0), \ g(U) \subset f(U). \tag{6}$$

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How to cite this paper: Oyekan, E. A., &Kehinde, R. (2018). Subordination results on certain new subclasses of analytic functions. *Confluence Journal of Pure and Applied Sciences (CJPAS),* 1(2), 56-63.

Vol. 1, No. 2, Nov. 2018
ISSN: 2616-1303 | **Web:**www.cjpas.fulokoja.edu.ng

Definition 3 (Subordinating factor sequence).

A sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be subordinating factor sequence if whenever $g(z)$ of the form (3) is analytic, univalent and convex in U, the subordination is given by

$$\sum_{k=1}^{\infty} a_k c_k z^k \prec g(z),\quad z \in U,\ a_1 = 1. \quad \text{(Wilf, 1961).}$$

For $f(z) \in A$, the following differential operator was introduced by (Salagean, 1983):

$$D^o f(z) = f(z), \quad D^1 f(z) = zf(z), \ldots, D^n f(z) = D(D^{n-1} f(z)),$$
$$(n \in \mathbf{N} = \{1,2,3,\ldots\}).$$

It is given that

$$D^n f(z) = z + \sum_{n=2}^{\infty} k^n a_n z^n \qquad (7)$$

Sequel to(7), (Raducanu and Orhan, 2010) defined the differential operator

$$D_{\lambda\mu}^m f(z) = z + \sum_{n=2}^{\infty} [1 + (\lambda\mu n + \lambda - \mu)(n-1)]\, a_n z^n \qquad (8)$$

as a generalization of (7).

Motivated by (8), (Oyekan, 2017a) also gives the following:

Definition 4: Let $n \in \mathbf{N}$; $\beta, \mu \in \mathbf{R}$ and $f(z)$ be as defined in (1). We denote by $D_{\beta,\mu}^n f(z)$ the linear operator defined by

$D_{\beta,\mu}^n f(z): A \to A;$

$$D_{\beta,\mu}^0 f(z) = f(z),$$
$$D_{\beta,\mu}^1 f(z) = D_{\beta\mu} f(z) = z(\mu - \beta) + z(1 + \beta - \mu) f'(z),$$
$$\vdots$$
$$D_{\beta,\mu}^n f(z) = D_{\beta,\mu}\left(D_{\beta,\mu}^{n-1} f(z)\right) \qquad (9)$$

where $1 \le \mu \le \beta$ and $n \in \mathbf{N} = \{1,2,\ldots\}$

We note that

$$D_{\beta,\mu}^n f(z) = z + \sum_{k=2}^{\infty} [k(1 + \beta - \mu)]^n a_k z^k \quad (10)$$

Definition 5 (Oyekan, 2017b): If $\alpha \ge 0, 1 \le \mu \le \beta$, then

$$U_{m,n}(\alpha,\beta,\mu,A,B) = \left\{f \in A: \left[\frac{D_{\beta,\mu}^m f(z)}{D_{\beta,\mu}^n f(z)} - \alpha\left|\frac{D_{\beta,\mu}^m f(z)}{D_{\beta,\mu}^n f(z)} - 1\right|\right] \prec \frac{1+Az}{1+Bz}\right.$$
$$\left. -1 \le B < A \le 1; m \in \mathbf{N}; n \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}; m > n; z \in U\right\}. \qquad (11)$$

Definition 5. Let $V_{m,n}^s(\alpha,\beta,\mu,A,B), s \in \mathbf{N}_0$ denote the subclass of A consisting of functions $f(z)$ which satisfy the following condition:

$$f(z) \in V_{m,n}^s(\alpha,\beta,\mu,A,B) \Leftrightarrow D^s f(z) \in U_{m,n}(\alpha,\beta,\mu,A,B), \qquad (12)$$

$$(-1 \le B < A \le 1;\ m \in \mathbf{N};\ n \in \mathbf{N}_0;\ m > n;\ z \in U)\ .$$

2 .0 MATERIAL AND METHODS

Basically the method to be used in this study is subordination principle via the techniques used earlier by (Srivastava and Attiya, 2004); (Attiya, 2005) and many others. We now state the lemmas needed to prove our results.

Lemma 6 (Wilf, 1961): The sequence $\{c_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if

$$Re\left\{1 + 2\sum_{k=1}^{\infty} c_k z^k\right\} > 0,\ z \in U. \qquad (13)$$

Lemma 7 (Oyekan, 2017b): A function $f(z)$ of the form (1) is in the class $U_{m,n}(\alpha,\beta,\mu,A,B)$ if

$$\sum_{k=2}^{\infty} \Omega(m,n,k,\alpha,\beta,\mu,A,B)|a_k| \le A - B \qquad (14)$$

where

$$\Omega(m,n,k,\alpha,\beta,\mu,A,B) = \left\{ [1 + \alpha(1 + |B|)]\left[(k(1+\beta-\mu))^m - (k(1+\beta-\mu))^n\right] + \left|B[k(1+\beta-\mu)]^m - A[k(1+\beta-\mu)]^n\right| \right\}$$

(15)

Lemma 8 (Oyekan, 2017b): A function $f(z)$ of the form (1) is in the class $V_{m,n}^{s}(\alpha,\beta,\mu,A,B)$ if

$$\sum_{k=2}^{\infty} k^{s}\Omega(m,n,k,\alpha,\beta,\mu,A,B)|a_k| \le A-B \quad (16)$$

where $\Omega(m,n,k,\alpha,\beta,\mu,A,B)$ is defined by (14).

In view of Lemma (7) and Lemma (8), it is natural to consider the classes $U_{m,n}^{*}(\alpha,\beta,\mu,A,B)$ and $V_{m,n}^{s*}(\alpha,\beta,\mu,A,B)$:

$$U_{m,n}^{*}(\alpha,\beta,\mu,A,B) = \left\{ f \in A: \sum_{k=2}^{\infty} \Omega(m,n,k,\alpha,\beta,\mu,A,B)|a_k| \le A-B \right\}, \quad (17)$$

$$V_{m,n}^{s*}(\alpha,\beta,\mu,A,B) = \left\{ f \in A: \sum_{k=2}^{\infty} k^{s}\Omega(m,n,k,\alpha,\beta,\mu,A,B)|a_k| \le A-B \right\}. \quad (18)$$

These classes consist of functions $f(z) \in A$ whose Taylor-Maclaurin coefficients satisfy the inequalities (14) and (16) respectively

Our main results in this paper are some subordination results associated with classes $U_{m,n}^{*}(\alpha,\beta,\mu,A,B)$ and $V_{m,n}^{s*}(\alpha,\beta,\mu,A,B)$.

3.0 RESULTS AND DISCUSSION

Unless otherwise mentioned, we assume in the remaining part of this paper that,
$-1 \le B < A \le 1$; $\alpha \ge 0$; $\beta,\mu \in \mathbb{R}$; $m \in \mathbb{N}$; $n \in \mathbb{N}_0$; $m > n$ and $z \in U$.

Theorem 9. (A subordination result associated with class $U_{m,n}^{*}(\alpha,\beta,\mu,A,B)$).
Let $f(z) \in U_{m,n}^{*}(\alpha,\beta,\mu,A,B)$, then

$$\frac{\Omega(2)}{2[\Omega(2)+\Delta]}(f*h)(z) \prec h(z) \quad (z \in U) \quad (19)$$

for every $h \in K$ and

$$Re(f(z)) > -\left(\frac{\Omega(2)+\Delta}{\Omega(2)}\right) \quad (20)$$

where

$$\Omega(2) = [1+\alpha(1+|B|)][(2(1+\beta-\mu))^{m} - (2(1+\beta-\mu))^{n}] + |B[2(1+\beta-\mu)]^{m} - A[2(1+\beta-\mu)]^{n}|$$

for $1 \le \mu \le \beta$, (21)

and

$$\Delta = A-B \quad (22)$$

The constant factor

$$\frac{\Omega(2)}{2[\Omega(2)+\Delta]} \quad (23)$$

in the subordination (19) cannot be replaced by a larger one.

Proof. Let $f(z) \in U_{m,n}^{*}(\alpha,\beta,\mu,A,B)$, and $z\sum_{k=2}^{\infty} c_k z^{k} \in K$. Then we have

$$\frac{\Omega(2)}{2[\Omega(2)+\Delta]}(f*h)(z) = \frac{\Omega(2)}{2[\Omega(2)+\Delta]}\left(z+\sum_{n=2}^{\infty} a_k c_k z^{k}\right) \quad (24)$$

Hence, by Definition 3 the subordination (19) will hold if,

$$\left\{\frac{\Omega(2)}{2[\Omega(2)+\Delta]}\right\}_{k=1}^{\infty} \quad (25)$$

is a subordinating factor sequence with $a_1 = 1$. In view of Lemma (6), this is equivalent to the following inequality:

$$Re\left\{1+2\sum_{k=1}^{\infty}\frac{\Omega(2)}{2[\Omega(2)+\Delta]}a_k z^{k}\right\} > 0; (z \in U) \quad (26)$$

Now since

$$\Omega(k) = [1+\alpha(1+|B|)][(k(1+\beta-\mu))^{m} - (k(1+\beta-\mu))^{n}] + |B[k(1+\beta-\mu)]^{m} - A[k(1+\beta-\mu)]^{n}|$$

is an increasing function of $k(k \geq 2)$ and $(m > n)$, we have

$$Re\left\{1+2\sum_{k=1}^{\infty}\frac{\Omega(2)}{2[\Omega(2)+\Delta]}a_k z^k\right\}$$

$= Re$

$$\left\{1+\frac{\Omega(2)}{[\Omega(2)+\Delta]}z+\frac{1}{[\Omega(2)+\Delta]}\sum_{k=2}^{\infty}\Omega(2)a_k z^k\right\} \quad (28)$$

$$\geq 1-\frac{\Omega(2)}{[\Omega(2)+\Delta]}r-\frac{1}{[\Omega(2)+\Delta]}\sum_{k=2}^{\infty}\Omega(k)|a_k|r^k$$

$$> 1-\frac{\Omega(2)}{[\Omega(2)+\Delta]}r-\frac{\Delta}{[\Omega(2)+\Delta]}r = 1-r > 0 \;(|z|=r<1),$$

where we have also made use of the assertion (14) of Lemma (7).

Next we show that

$$Re(f(z)) > -\left(\frac{\Omega(2)+\Delta}{\Omega(2)}\right).$$

To do this, we take $h=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k \in K$ in (19) to get

$$\frac{\Omega(2)}{\Omega(2)+\Delta}\,\pi\,\frac{z}{1-z} \quad (29)$$

i.e.,

$$Re\left(\frac{\Omega(2)}{\Omega(2)+\Delta}\right) > Re\left(\frac{z}{1-z}\right). \quad (30)$$

Since

$$Re\left(\frac{z}{1-z}\right) > -\frac{1}{2}, \quad |z|<r, \quad (31)$$

we have

$$\frac{\Omega(2)}{\Omega(2)+\Delta}Re(f(z)) > -\frac{1}{2} \quad (32)$$

Therefore,

$$Re(f(z)) > -\left(\frac{\Omega(2)+\Delta}{\Omega(2)}\right) \quad (33)$$

(27)

which is (20).

To show the sharpness of the constant $\frac{\Omega(2)}{2[\Omega(2)+\Delta]}$, we take $h(z)=\frac{z}{1-z}$ and $f(z)=f_1(z)$ in (19) where

$$f_1(z)=z-\frac{\Delta}{\Omega(2)}z^2=\frac{z\Omega(2)-\Delta z^2}{\Omega(2)} \quad (34)$$

Such that

$$\frac{z\Omega(2)-\Delta z^2}{2(\Omega(2)+\Delta)}\,\pi\,\frac{z}{1-z}. \quad (35)$$

Since it is known that $|\mathrm{Re}(z)| \leq |z|$, we need to show that

$$\min\left\{\mathrm{Re}\left(\frac{z\Omega(2)-\Delta z^2}{2(\Omega(2)+\Delta)}\right);\quad z\in U\right\}=-\frac{1}{2}. \quad (36)$$

Now

$$\left|\mathrm{Re}\left(\frac{z\Omega(2)-\Delta z^2}{\Delta}\right)\right| \leq \left|\frac{z\Omega(2)-\Delta z^2}{2(\Omega(2)+\Delta)}\right|$$

$$=\frac{|z\Omega(2)-\Delta z^2|}{|2(\Omega(2)+\Delta)|} \leq \frac{|z||\Omega(2)+|\Delta||z|^2}{2(\Omega(2)+\Delta)} \quad (37)$$

$$=\frac{\Omega(2)+\Delta}{2(\Omega(2)+\Delta)}=\frac{1}{2} \quad (|z|=1).$$

This implies that

$$\left|\mathrm{Re}\left(\frac{z\Omega(2)-\Delta z^2}{2(\Omega(2)+\Delta)}\right)\right| \leq \frac{1}{2} \quad (38)$$

i.e.,

$$-\frac{1}{2} \leq \mathrm{Re}\left(\frac{z\Omega(2)-\Delta z^2}{2(\Omega(2)+\Delta)}\right) \leq \frac{1}{2} \quad (39)$$

Hence

$$\min\left\{\operatorname{Re}\left(\frac{z\Omega(2)-\Delta z^2}{2(\Omega(2)+\Delta)}\right);\quad z\in U\right\}=-\frac{1}{2}. \quad (40)$$

This completes the proof of Theorem (9)

Then

$$\frac{(1+\alpha(1+|B|))(2^m-2^n)+|B(2^m)-A(2^n)|}{2[(1+\alpha(1+|B|))(2^m-2^n)+|B(2^m)-A(2^n)|]+(A-B)}(f*h)(z) \prec h(z) \quad (z\in U). \quad (41)$$

for every $h\in K$

and

$$Re(f(z)) > -\left(\frac{2[(1+\alpha(1+|B|))(2^m-2^n)+|B(2^m)-A(2^n)|]+(A-B)}{(1+\alpha(1+|B|))(2^m-2^n)+|B(2^m)-A(2^n)|},\quad z\in U\right) \quad (42)$$

The constant factor

$$\frac{(1+\alpha(1+|B|))(2^m-2^n)+|B(2^m)-A(2^n)|}{2[(1+\alpha(1+|B|))(2^m-2^n)+|B(2^m)-A(2^n)|]+(A-B)} \quad (43)$$

in the subordination (41) cannot be replaced by a larger one.

Remark 2:

Taking $\beta=\mu=1$ and $1^m=1^n=1$ for all $m\in\mathbf{N}$; $n\in\mathbf{N}_0$; $m>n$ in Theorem (9) we have the following:

Corollary 10. Let $f(z)\in U^{*}_{m,n}(\alpha,1,1,A,B)$.

The result in the **Corollary 10** is the result obtained by (Aouf et al., 2012 **[Theorem 3]**)

The proof of the following subordination result is similar to that of Theorem (9).

By making use of Lemma (8) in place of Lemma (7) and let $s=0$, we get the desired result. Therefore, we omit the analogous details involved.

Theorem 11. (A subordination result associated with the class $V^{s*}_{m,n}(\alpha,\beta,\mu,A,B)$).

Let $f(z)\in V^{s*}_{m,n}(\alpha,\beta,\mu,A,B)$, then

$$\frac{2^s\Omega(2)}{2[2^s\Omega(2)+\Delta]}(f*h)(z) \prec h(z) \quad (z\in U) \quad (44)$$

$$Re(f(z)) > -\left(\frac{2^s\Omega(2)+\Delta}{2^s\Omega(2)}\right) \quad (45)$$

where

$$\Omega(2)=[1+\alpha(1+|B|)][(2(1+\beta-\mu))^m-(2(1+\beta-\mu))^n]+\left|B[2(1+\beta-\mu)]^m-A[2(1+\beta-\mu)]^n\right| \quad (46)$$

and

$$\Delta=A-B \quad (47)$$

The constant factor

$$\frac{2^s\Omega(2)}{2[2^s\Omega(2)+\Delta]} \quad (48)$$

in the subordination (44) cannot be replaced by a larger one.

By suitably specializing the various parameters involve in Theorem (9), we get corresponding subordination results for certain known subclasses and new one.

(i) When $B=-1$, $A=1$, $m=1$, $n=0$, $\beta=\mu=1$ and $\alpha=0$ $(\alpha\geq 0)$, we have

Corrollary 12. Let $f(z)\in U^{*}_{1,0}(0,1,1,1,-1)$.

Then

$$\frac{1}{3}(f*h)(z) \prec h(z) \quad (z\in U)$$

for every $h\in K$ and

$$\operatorname{Re}(f(z)) > \frac{3}{2},\quad z\in U.$$

The constant factor $\frac{1}{3}$ cannot be replaced by a larger one. This is due to (Sukhjit, 2000) and (Selvaraj and Karthikeyan, 2008).

(ii) When $B=0,\ A=1,\ m=1,\ n=0,\ \beta=\mu=1$ and $\alpha=2\quad(\alpha\geq 0)$, we have

Corollary 13. Let $f(z)\in U^{*}_{1,0}(2,1,1,1,0)$. Then

$$\frac{2}{5}(f*h)(z)\pi\ h(z)\quad (z\in U)$$

for every $h\in K$ and

$$\mathrm{Re}(f(z))>\frac{5}{4},\quad z\in U.$$

The constant factor $\frac{2}{5}$ cannot be replaced by a larger one. This result was obtained by (Frasin, 2006) and (Selvaraj and Karthikeyan, 2008).

(iii) Taking $A=1-2\lambda\quad(0\leq\lambda<1)$, $\beta=\mu=1$ and $B=-1$ in Theorem (9), we correct the result obtained by (Srivastava and Eker, 2008 [Theorem 1]);

(iv) Taking $A=1-2\lambda\quad(0\leq\lambda<1)$, $\beta=\mu=1$, $m=n+1(n\in\aleph_0)$ and $B=-1$ in Theorem (9), we obtain the result obtained by (Aouf et al., 2010 [Corollary 4]);

(v) Taking $A=1-2\lambda\quad(0\leq\lambda<1)$, $m=1,\ n=0,\ \beta=\mu=1\ \ \alpha=1$ and $B=-1$ in Theorem (9), we obtain the result obtained by (Aouf et al., 2010 [Corollary 1]);

(vi) Taking $A=1-2\lambda\quad(0\leq\lambda<1), m=2,\ n=1,\ \beta=\mu=1\quad\alpha=1$ and $B=-1$ in Theorem (9), we obtain the result obtained by (Aouf et al., 2010 [Corollary 2]);

(vii) Taking $A=1, m=2,\ n=1,\ \beta=\mu=1$ and $B=-1$ in Theorem (9), we obtain the result obtained by (Aouf et. al.,2010 [Corollary 3]);

(viii) Taking $A=1-2\lambda\quad(0\leq\lambda<1), m=1,\ n=0,\ \beta=\mu=1$ and $B=-1$ in Theorem (9), we obtain the obtained by (Frasin, 2006 [Corollary 2.2]);

(ix) Taking $A=1-2\lambda\quad(0\leq\lambda<1), m=2,\ n=1,\ \beta=\mu=1$ and $B=-1$ in Theorem (9), we obtain the result obtained by (Frasin, 2006 [Corollary 2.5]);

(x) Taking $A=1-2\lambda\quad(0\leq\lambda<1), m=1,\ n=0,\ \beta=\mu=1\quad\alpha=0$ and $B=-1$ in Theorem (9), we obtain the result obtained by (Frasin, 2006 [Corollary 2.3]);

(xi) Taking $A=1-2\lambda\quad(0\leq\lambda<1), m=2,\ n=0,\ \beta=\mu=1\quad\alpha=0$ and $B=-1$ in Theorem (9), we obtain the result obtained by (Frasin, 2006 [Corollary 2.6]);

(xii) Taking $A=1,\ m=2,\ n=1,\ \beta=\mu=1\quad\alpha=0$ and $B=-1$ in Theorem (9), we obtain the result obtained by (Frasin, 2006 [Corollary 2.7]);

Also, we observe that the classes $S^{*}(A,B)$ and $K^{*}(A,B)$, defined as follows yield results which are special cases of our main result:

$$U_{1,0}(0,1,1,A,B)=S^{*}(A,B)=\left\{f\in A:\frac{zf'(z)}{f(z)}\ \pi\ \frac{1+Az}{1+Bz}\right\},\qquad(49)$$

$(-1\leq B<A\leq 1;\ z\in U)$

$$U_{1,0}(0,1,1,A,B)=K^{*}(A,B)=\left\{f\in A:\ 1+\frac{zf''(z)}{f'(z)}\ \pi\ \frac{1+Az}{1+Bz}\right\},\qquad(50)$$

$(-1\leq B<A\leq 1;\ z\in U)$

(Janowski, 1973) and (Padmanabhan and Ganesan, 1988).

Furthermore we note that

$$U_{m,n}(0,1,1,A,B)=U(m,n;A,B)=\left\{f\in A:\ 1+\frac{D^{m}f(z)}{D^{n}f(z)}\ \pi\ \frac{1+Az}{1+Bz}\right\},\qquad(51)$$

$$(-1\leq B<A\leq 1;\ m\in\mathbf{N};\ n\in\mathbf{N}_0;\ m>n).$$

Vol. 1, No. 2, Nov. 2018
ISSN: 2616-1303 | **Web:** *www.cjpas.fulokoja.edu.ng*

Following from (49), (50) and (51) we state the following subordination results:

(i) Putting $\alpha = n = 0$; $\beta = \mu = m = 1$ and $1^m = 1^n = 1$ for all $m \in \mathbb{N}$; $n \in \mathbb{N}_0$; $m > n$ in Theorem 9 , we have

Corollary 14. Let $f(z) \in S^*(A,B)$. Then

$$\frac{1+|2B-A|}{2[1+|2B-A|+(A-B)]}(f*h)(z) \pi\, h(z) \quad (z \in U) \qquad (52)$$

for every $h \in K$ and

$$\operatorname{Re}(f(z)) > -\frac{1+|2B-A|+(A-B)}{1+|2B-A|}, \quad z \in U.$$

The constant factor

$$\frac{1+|2B-A|}{2[1+|2B-A|+(A-B)]}$$

in the subordination (52) cannot be replaced by a larger one.

(ii) Putting $\alpha = 0$, $n = 1$, $\beta = \mu = 2$ and $m = 1$ in Theorem 9 we have

Corollary 15. Let $f(z) \in K^*(A,B)$. Then

$$\frac{1+|2B-A|}{2+2|2B-A|+(A-B)}(f*h)(z) \pi\, h(z) \quad (z \in U) \qquad (53)$$

for every $h \in K$ and

$$\operatorname{Re}(f(z)) > -\frac{2+|2B-A|+(A-B)}{2(1+|2B-A|)}, \quad z \in U.$$

The constant factor

$$\frac{1+|2B-A|}{2+2|2B-A|+(A-B)}$$

in the subordination (53) cannot be replaced by a larger one.

(i) Putting $\alpha = 0$, in Theorem 9 with $\beta = \mu = 1$ and $1^m = 1^n = 1$ for all $m \in \mathbb{N}$; $n \in \mathbb{N}_0$; $m > n$ we have

Corollary 16. Let $f(z) \in K^*(A,B)$. Then

$$\frac{[2^m-2^n]+|B(2^m)-A(2^n)|}{[2^m-2^n]+|B(2^m)-A(2^n)|+(A-B)}(f*h)(z) \prec h(z), \quad (z \in U) \qquad (54)$$

for every $h \in K$ and

$$Re(f(z)) > -\frac{[2^m-2^n]+|B(2^m)-A(2^n)|+(A-B)}{[2^m-2^n]+|B(2^m)-A(2^n)|}, \quad z \in U$$

The constant factor

$$\frac{[2^m-2^n]+|B(2^m)-A(2^n)|}{[2^m-2^n]+|B(2^m)-A(2^n)|+(A-B)}$$

in the subordination (54) cannot be replaced by a larger one.

CONCLUSION

In this work, we defined two subclasses of analytic functions and derive subordination results for them. It was discovered that some earlier known results are special cases of our results. Thus our results extend the earlier ones. Furthermore, some of our results are new

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